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Construction of Positive Exact (2 + 1)-Dimensional Shock Wave Solutions for Two Discrete Boltzmann Models

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It is proved that $(2+1)$-dimensional (space x, y; time t) positive exact shock wave solutions of two discrete Boltzmann models exist. For each density N_i, these solutions are linear combinations of three similarity shock waves, $N_i = n_{0i} + \sum_j n_{ji}/[1 + d_j \exp(\tau_j y + \gamma_j x + \rho_j t]$, $j = 1, 2, 3$. Two models with four independent densities are investigated: the square discrete-velocity Boltzmann model and the model with eight velocities oriented toward the eight corners of a cube. *The positivity problem for the densities is nontrivial.* Two classes of solutions are considered for which the two first similarity shock wave components depend on only one spatial dimension, $\gamma_j = \text{const} \cdot \tau_j$, $j = 1, 2$. For the positivity, if $\tau_1 \tau_2 > 0$, it is sufficient to prove that the 16 asymptotic shock limits n_{0i}, $n_{0i} + n_{3i}$, $\sum_{j=0}^{2} n_{ji}$, $\sum_{j=0}^{3} n_{ji}$ are positive. The density solutions are built up with five arbitrary parameters and we prove that there exist subdomains of the arbitrary parameter space in which the 16 shock limits are positive. We study numerically two explicit shock wave solutions. We are interested in the movement of the shock front when the time is growing and in the possible appearance of bumps. In the space, at intermediate times, these bumps represent populations of particles which are larger than at initial time or at equilibrium time.

KEY WORDS: Kinetic theory; discrete Boltzmann models; shock waves; exact solutions of nonlinear equations.

1. INTRODUCTION

There has been much study of discrete Boltzmann models, where the velocities can only take the discrete values $\mathbf{v}_i$, $|\mathbf{v}_i| = 1$, in the hope of finding useful results for both kinetic theory and fluid mechanics. Since the popular

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Broadwell[1] model, which provided for the first time an explicit solution of an infinite-strength shock, many others have been proposed.[2] To each velocity $\mathbf{v}_i$ is associated a density N_i and for the N_i with two spatial coordinates we must consider models velocities in a plane or in a three-dimensional space.

In $(1+1)$ dimensions (space x, time t), the exact solutions are the sums of two similarity shock waves,[3] and four classes of different solutions are known: (1) shock waves,[3] (2) periodic propagating solutions,[3] (3) solutions that are periodic in space but nonpropagating in time,[3-5] (4) densities relaxing toward nonuniform Maxwellians.[3]

In the $(2+1)$-dimensional space, exact solutions are missing. The discovery of exact two-spatial-dimensional solutions could help toward the theoretical understanding of these models. From the physical point of view it is clear that $(2+1)$-dimensional solutions are more realistic than $(1+1)$-dimensional solutions. As we shall see, the construction of such solutions is relatively simple; *the great difficulty is the positivity condition.*

The aim of this paper is twofold. First, to give a rigorous proof of the existence of positive shock waves, and second, to explore some physical aspects of these solutions.

We consider two models; the first is the square-velocity model[2-6] attributed to Maxwell with $\mathbf{v}_1$ and $\mathbf{v}_3$ along the positive x and y axes, $\mathbf{v}_1+\mathbf{v}_2=\mathbf{v}_3+\mathbf{v}_4=0$, leading to the equations

$$\begin{aligned} N_{1t}+N_{1x}&=N_{2t}-N_{2x}=-N_{3t}-N_{3y}=-N_{4t}+N_{4y}\\ &=aN_3N_4-N_1N_2, \qquad a>0 \end{aligned} \tag{1.1}$$

The second model is cubic,[7] with eight velocities oriented toward the eight corners of a cube, with four independent N_i ($N_6=N_1$, $N_5=N_2$, $N_8=N_3$, $N_7=N_4$), and the equations reduce to (1.1) with the change of variables $(x+y)/2\to x$, $(y-x)/2\to y$. The total mass is $M=\sum_i N_i$ with $i=1,\dots,4$ for the first model and $i=1,\dots,8$ for the second one. Both mass and momentum conservation laws hold. For instance, $M_t+\partial_x J_{(x)}+\partial_y J_{(y)}=0$ with components $J_{(x)}=N_1-N_2$ and $J_{(y)}=N_3-N_4$ for the momentum $\mathbf{J}$. For $a>0$ but $a\neq 1$ the microreversibility is violated. Introducing[6] the relative entropy $H=\sum N_i \log(N_i/\alpha_i)$, $\alpha_i>0$, $\alpha_1\alpha_2=a\alpha_3\alpha_4$, we find from (1.1), as usual, $H_t+(\partial_x\cdots+\partial_y\cdots)\,H\leqslant 0$.

The similarity shock waves are

$$N_i=n_{0i}+n_i/D_i, \qquad D=1+d\exp(\tau y+\gamma x+\rho t) \tag{1.2}$$

where n_{0i}, n_i, τ, γ, ρ, $d>0$ are constants, while the $(2+1)$-dimensional solutions are simply the sums of such solutions:

$$N_i = n_{0i} + \sum_j n_{ji}/D_j, \qquad D_j = 1 + d_j \exp(\tau_j y + \gamma_j x + \rho_j t), \qquad d_j > 0 \tag{1.3}$$

Substituting (1.3) into (1.1) and writing that the coefficients of D_j^{-1}, D_j^{-2}, const, $(D_m D_p)^{-1}$, $m \neq p$, are zero, we find

$$\begin{gathered} n_{j1}(\rho_j+\gamma_j) = n_{j2}(\rho_j-\gamma_j) = -n_{j3}(\rho_j+\tau_j) = n_{j4}(\tau_j-\rho_j) \\ = a n_{j3} n_{j4} - n_{j1} n_{j2} = -a(n_{03} n_{j4} + n_{04} n_{j3}) + n_{01} n_{j2} + n_{02} n_{j1} \\ a n_{03} n_{04} = n_{01} n_{02} \\ a(n_{m3} n_{p4} + n_{m4} n_{p3}) = n_{m1} n_{p2} + n_{m2} n_{p1}, \qquad m \neq p \end{gathered} \tag{1.4}$$

Neglecting the $m \neq p$ relations, we see that the others represent the conditions for each jth component to be similarity solutions. However, (1.1) is not a linear system; in order for the sum to be a solution we must have supplementary conditions [the last of (1.4)]. For a sum of N similarity components we have $N(N-1)/2$ supplementary conditions. Even if the constraints (1.4) are compatible, the solutions are physically acceptable only if they lead to positive N_i.

In the sequel we consider a superposition of three similarity components with 25 parameters and 19 relations, leaving six arbitrary parameters. Although solutions satisfying (1.4) are easily found with the help of the computer, I was unable to find any positive solution. This means that we must understand the mathematical structure of the positivity constraints. Recently,[8] for the simplest solutions (1.3), an analytic proof of the existence of positive solutions was shown to be possible. These solutions relax toward nonuniform Maxwellians; unfortunately, they are physically poor, because their total masses are constants.

The aim of this paper is to prove analytically that positive $(2+1)$-*dimensional shock waves exist.* What are the positivity constraints? In one spatial coordinate x we only have two asymptotic shock liits[3] when $|x| \to \infty$ for each N_i at $t=0$. If these limits are positive, we can manage the d_j so that $N_i > 0$ for any x value. In (1.3), let us define $D_j = 1 + d_j \exp X_j$, $X_3 = \text{const}_1 \cdot X_1 + \text{const}_2 \cdot X_2$. In the X_1, X_2 plane at $t=0$ (or x, y plane) six asymptotic shock limits exist for each N_i (for instance, if the axis $X_3 > 0$ is in the first X_1, X_2 quadrant, we find n_{0i}, $n_{0i}+n_{1i}$, $n_{0i}+n_{2i}$, $n_{0i}+n_{2i}+n_{3i}$, $n_{0i}+n_{1i}+n_{3i}$, $\sum n_{ji}$, $j=0,\ldots,3$). Unfortunately, (1.4) is much too complicated to be solved analytically and we choose a simpler situation.

In this paper we assume that the first two $j=1, 2$ components depend upon only one coordinate, $y + \text{const} \cdot x$ at $t=0$:

$$D_j = 1 + d_j \exp[\tau_j(y+\mu x) + \rho_j t], \qquad \gamma_j = \tau_j \mu, \qquad j = 1, 2 \tag{1.3'}$$

In the x, y plane only four asymptotic shock limits exist, depending on the $\tau_1\tau_2$ sign:

$$\begin{aligned}
&\tau_1\tau_2>0:\quad n_{0i},\qquad \Sigma_i^{03}=n_{0i}+n_{3i},\qquad \Sigma_i^2=\sum_{j=0}^{2} n_{ji},\qquad \Sigma_i^3=\sum_{j=0}^{3} n_{ji}\\
&\tau_1\tau_2<0:\quad n_{0i}+n_{ji}+n_{3i},\qquad n_{0i}+n_{ji},\qquad j=1,2
\end{aligned}\tag{1.5}$$

In Appendix A it is shown that if the four asymptotic shock limits (1.4) are positive, then we can choose the d_j such that the N_i are positive.

In Sections 2 and 3 we prove (see Appendices B and C for the details) that in a space of five arbitrary parameters, from which we reconstruct all the n_{0i}, n_{ji}, τ_j, γ_j, ρ_j parameters of the N_i, there exist subdomains where the 16 Σ_i (corresponding to $\tau_1\tau_2>0$) shock limits are positive. If we define $z_j=n_{j4}/n_{j3}$, $j=1,2$, $P=z_1z_2$, and $S=z_1+z_2$, the chosen five arbitrary parameters are

$$(P, S, n_{0i}>0,\ i=1,2,3)\tag{1.6}$$

The mathematical structure of these shock limits, allowing an analytical positivity study, is provided by a factorization property. All Σ_i are linear combinations of the four n_{0i} with P, S-dependent coefficients. Further, they can be written as second-degree n_{03} polynomials:

$$n_{03}\Sigma_i=\Omega_i(n_{03}-n_{01}A_k)(n_{03}-n_{02}A_{k'})\tag{1.7}$$

with P, S-dependent coefficients (see Tables I and II). For each Σ_i we seek the n_{03} interval in which Σ_i is positive and study the intersections of these 12 intervals. Further, we must compare the roots of the Σ_i and we find that the intersection is not empty if the ratio n_{01}/n_{02} has either a P, S-dependent lower or upper bound. All these calculations are tedious; however, invariance properties allow us to reduce the task with the possibility of finding Σ_2 from Σ_1 and Σ_4 from Σ_3.

(i) From (1.1) we see that $x\leftrightarrow -x$ is equivalent to $N_1\leftrightarrow N_2$. For $\Sigma_1\to\Sigma_2$ we change $n_{01}\leftrightarrow n_{02}$ and for $j=1,2$, $n_{j1}\leftrightarrow n_{j2}$ [or $\mu\leftrightarrow -\mu$ from (1.3)]. For the exchange $n_{31}\leftrightarrow n_{32}$ we have introduced a P, S-dependent parameter y_3 in the formalism and $y_3=n_{31}/n_{32}$ becomes $1/y_3$.

(ii) For $\Sigma_3\leftrightarrow\Sigma_4$ we change $n_{03}\leftrightarrow n_{04}$ and $n_{j3}\leftrightarrow n_{j4}$. From the definition of z_j, this is equivalent for $j=1,2$ to $z_j\to 1/z_j$ or $P\to 1/P$ and $S\to S/P$.

In Section 2 we choose the simplest case, $\mu=0$ in (1.3′), or the $j=1,2$ components only y dependent at $t=0$. This is a pedagogical example for which the mathematical machinery is tractable. The final result, Theorem 1, gives the explicit P, S domain, the $n_{01}/n_{02}(P, S)$ upper bound, and the $n_{03}(P, S)$ interval for which all Σ_i are positive. The price to be paid for this relative simplicity is that the microreversibility parameter a is P, S dependent and $a < 1/3$, which excludes $a = 1$.

In Section 3 we look at the more general case where the two first components are $y + \mu(P, S)x$ dependent at $t = 0$. The mathematical analysis is more complicated than in Section 2, but we find positive solutions satisfying the microreversibility ($a = 1$). We give the expressions of the Σ_i in terms of the arbitrary parameters; however, for the positivity we restrict the study to the case $S = -2(P+1)$. In Theorems 2 and 3 we find two subdomains of the arbitrary parameter space in which all Σ_i are positive.

In Section 4 we choose two examples satisfying Theorems 1 and 2, leading to $N_i > 0$, and construct their total masses $M = \sum N_i$. For both examples we study numerically the equidensity lines $M = \text{const}$ at $t = 0$ and the relaxation curves N_i, M when the time is growing. For M the four asymptotic shock limits become

$$m_0 = \sum_i n_{0i}, \qquad \Sigma^{03} = \sum_i \Sigma_i^{03}, \qquad \Sigma^2 = \sum_i \Sigma_i^2, \qquad \Sigma^3 = \sum_i \Sigma_i^3 \tag{1.8}$$

leading to a physical structure more interesting than in one spatial coordinate. These shock limits represent plateaus in the spatial coordinate plane separated by the shock domain. We find the two highest plateaus in the upstream domain, while the two lowest belong to the downstream domain. We look at the possible ways to decrease equidensity lines to link the highest plateau to the lowest one. We find two different scenarios. First, the equidensity lines decrease continuously from the highest plateau, cross the shock domain, and spread out into the downstream domain. In the second scenario the upstream and downstream domains are completely isolated by the shock front. A bump is always present in the shock domain. Looking at the displacement of the equidensity lines when the time is varying, the second scenario can appear. It can happen that for intermediate times, populations of particles larger than at initial time or at equilibrium exist. Physically, this can be explained by a compression of particles, while mathematically we explain this effect by a shifting of the d_j parameters in (D_j) to $d_j \exp(\rho_j t)$. We study also the movement of the shock when the time is growing.

2. MODELS WITH TWO SIMILARITY COMPONENTS WITH ONLY A y SPATIAL DEPENDENCE

We study the $(2+1)$-dimensional solutions

$$N_i = n_{0i} + \sum_{j=1}^{3} n_{ji}/D_j, \qquad D_j = 1 + d_j \exp(\tau_j y + \gamma_j x + \rho_j t)$$
$$\gamma_1 = \gamma_2 = 0, \qquad i = 1, \ldots, 4 \tag{2.1}$$

The first two n_{ji}/D_j, $j = 1, 2$, components are x independent. Our aim is to prove analytically that there exists a class of solutions N_i such that the asymptotic shock limits Σ_i

$$\Sigma_i^0 = n_{0i}, \qquad \Sigma_i^2 = \sum_{j=0}^{2} n_{ji}, \qquad \Sigma_i^{03} = n_{0i} + n_{3i}, \qquad \Sigma_i^3 = \sum_{j=0}^{3} n_{ji} \tag{2.2}$$

are positive. All details and proofs are given in Appendix B; here we quote only the main results. First we write down the expressions of the parameters of the solutions N_i as functions of five arbitrary parameters. Second, we determine the Σ_i in terms of these arbitrary parameters. Finally, in the five-dimensional parameter space we find a subspace where the Σ_i as well as $\tau_1 \tau_2$ are positive.

2.1. Solutions N_i (Appendices B.1, B.2)

There exist 19 relations among the 23 parameters n_{0i}, n_{ji}, τ_j, ρ_j, γ_3. However, since the microreversibility parameter $a > 0$ is not fixed, one supplementary parameter is left. The solutions depend upon five arbitrary parameters, from which we must express all the others.

We follow the same method as for the previous construction of exact $(1+1)$-dimensional solutions.[3] For each jth component we define a scaling parameter which is the ratio of two well-defined n_{ji}. It turns out that all the other ratios n_{jk}/n_{ji} are functions of these three scaling parameters. Further, one of these scaling parameters can be expressed as a function of the other two and we are left with only two of these scaling parameters. We obtain the n_{ji} as linear combinations of the four n_{0i} with coefficients that are functions of the two remaining scaling parameters. Finally, the τ_j, ρ_j, γ_3 are functions of the n_{ji}. We define two scaling parameters $z_j = n_{j3}/n_{j4}$ and choose for the five arbitrary parameters

$$(P = z_1 z_2,\ S = z_1 + z_2;\ n_{0i},\ i = 1, 2, 3) \tag{2.3}$$

The microreversibility parameter a is P, S dependent, while n_{04} and all other parameters belonging to the first two $j=1, 2$ components depend upon the five arbitrary ones:

$$a = 8P/S(S+P+1), \qquad n_{04} = n_{01}n_{02}/an_{03} \tag{2.4}$$

$$n_{j3} = 2\{P[n_{03}(1+z_j) + 2(n_{01}+n_{02})/a] + n_{04}(z_i+P)\}$$
$$\times [(z_j - z_i)(z_j - P)]^{-1}, \qquad i \neq j \tag{2.5}$$

$$n_{j4} = z_j n_{j3}, \qquad n_{j1} = n_{j2} = -2z_j n_{j3}/(1+z_j), \qquad j=1, 2 \tag{2.6}$$

$$\begin{gathered} 2\tau_j z_j = (z_j - 1)[a(n_{03}z_j + n_{04}) + 2z_j(n_{01}+n_{02})/(1+z_j)] \\ \rho_j = -\tau_j n_{j3}/(n_{j1}+n_{j3}), \qquad j=1, 2 \end{gathered} \tag{2.7}$$

For the third component we introduce a third scaling parameter y_3, which is S, P dependent:

$$\begin{gathered} y_3 = n_{31}/n_{32}, \qquad (1+y_3)^2 = 4(P+1)\,y_3/S \\ y_3^{\pm} = -B' \pm (B'^2 - 1)^{1/2}, \qquad B' = 1 - 2(1+P)/S \\ n_{32}(P+1-S) = (P+1+S)(n_{02} + n_{01}/y_3) + 2(n_{03}P + n_{04})(1+1/y_3) \\ n_{33} = -y_3(1+P)n_{32}/P(1+y_3), \qquad n_{31} = y_3 n_{32}, \qquad n_{34} = Pn_{33} \\ \rho_3 n_{33} n_{34} = (n_{33}+n_{34})(n_{31}n_{32} - an_{33}n_{34})/2 \\ \tau_3(n_{33}+n_{34}) = \rho_3(n_{34} - n_{33}) \\ \gamma_3(n_{32}+n_{31}) = \rho_3(n_{32} - n_{31}) \end{gathered} \tag{2.8}$$

We have constructed a five-parameter family of N_i solutions. However, the physically acceptable solutions must have $N_i > 0$, and if $\tau_1\tau_2 > 0$, it is sufficient that the 16 shock limits Σ_i given by (2.2) are positive. The four conditions $n_{0i} > 0$ are easily satisfied if we choose $n_{0i} > 0$ for $i = 1, 2, 3$ and P, S values such that a is positive in (2.4).

2.2. Analytic Expressions for the Σ_i (Appendix B.3)

First we remark that all the n_{ji} written down above are *linear combinations of the four* n_{0i}, so that the same property holds for the 12 Σ_i. Second, from the relation (2.4) for n_{04} we see that $n_{03}\Sigma_i$ will be *second-degree polynomials* in n_{03} with coefficients that are functions of P, S, n_{01}, and n_{02}. However, there exist invariance properties:

(i) For $i = 1, 2$ the quadratic relations are

$$\begin{gathered} n_{03}\Sigma_1 = \Omega_1(n_{03} - n_{01}A_k)(n_{03} - n_{02}A_{k'}) \\ n_{03}\Sigma_2 = \Omega_2(n_{03} - n_{01}\tilde{A}_{k'})(n_{03} - n_{02}\tilde{A}_k) \end{gathered} \tag{2.9}$$

with Ω_i, A_k, and $A_{k'}$ functions of P, S, and eventually of y_3. In this last case $\tilde{A}_k = A_k(y_3 \to 1/y_3)$. From the relations $n_{j1} = n_{j2}$, $j = 1, 2$, $n_{31}/n_{32} = y_3$ we deduce that $1 \leftrightarrow 2$ if both $n_{01} \leftrightarrow n_{02}$ and $y_3 \leftrightarrow 1/y_3$.

(ii) Similarly we can obtain $3 \leftrightarrow 4$ if we exchange both $n_{03} \leftrightarrow n_{04}$ and $z_j \leftrightarrow 1/z_j$ or $P \leftrightarrow 1/P$, $S \leftrightarrow S/P$.

(iii) Are there relations between the Σ_i of the first family $i = 1, 2$ and those $i = 3, 4$ of the second one? As we show now, they share common roots $n_{03} = n_{0j} A_k$ or $n_{0j} \tilde{A}_k$. The condition [see (B.32)] for a common Σ_1^2, Σ_3^2 root is

$$n_{01}(1 + P + S) + 4n_{04} = 0 \quad \text{or} \quad n_{04} \to n_{01} n_{02}/a n_{03}, \quad n_{03}/n_{02} = -S/2P = A_2 \tag{2.10}$$

$A_2 n_{02}$ being one zero of Σ_1^2, it is also a zero of Σ_3^2. From the $3 \leftrightarrow 4$ symmetry in (ii), we deduce that $n_{03}/n_{04} = -(P + S + 1)/4P = A_1$ is the common zero of Σ_1^2, Σ_4^2. In the same way, with the symmetry $1 \leftrightarrow 2$ of (i), we find that $n_{03} = n_{01} A_2$ is a zero common to Σ_2^2, Σ_3^2, while $n_{02} A_1$ is common to Σ_2^2, Σ_4^2. For Σ_2^{03}, Σ_3^{03} the possible root is

$$n_{04} 8(P + 1) + n_{01}(S + P + 1)\, S(1 + 1/y_3) = 0$$

or

$$n_{03}/n_{02} = \tilde{A}_3 = -(P + 1)/P(1 + 1/y_3) \tag{2.11}$$

and is in fact the common root. With the symmetries $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, we deduce that $n_{01} A_3$ is a common zero of Σ_1^{03}, Σ_3^{03}; $n_{01} \tilde{A}_4$ is common to Σ_2^{03}, Σ_4^{03}; while $n_{02} A_4$ is common to Σ_1^{03}, Σ_4^{03}.

Finally, for each Σ_i family there exist only four different roots and this result simplifies the positivity study of the Σ_i.

2.3. Sufficient Conditions So That All Σ_i Are Positive (Appendices B.4 and B.5 and Table I)

In the five-dimensional parameter space, the analytic determination of a subspace in which all the Σ_i are positive seems untractable. For each of the 12 second-degree n_{03} polynomials, we must check both the sign of the coefficient of n_{03}^2 and the location of the two roots $n_{0j} A_k$ or $n_{0j} A_{k'}$ and determine the intervals of n_{03} in which $\Sigma_i > 0$. Afterward we must check that the intersections of these 12 intervals are not empty. Fortunately, scaling parameters exist which simplify the discussion. Practically, the study of three parameters will be important.

We introduce a new arbitrary parameter s, a function of both P and S, and which replaces S. We also define new functions deduced with the factorization of trivial factors:

$$\begin{gathered} s = -S/(P+1), \qquad B_i = A_i P/(P+1), \qquad B_i = A_i P/(P+1) \\ \bar{n}_{03} = n_{03} P/(P+1), \qquad \bar{\Sigma}_i n_{03} = \Sigma_i (s+1) \end{gathered} \tag{2.12}$$

In Table I the 12 $\bar{\Sigma}_i$ are written down as second-degree $\bar{n}_{03}$ polynomials with roots $n_{0j} B_k$ or $n_{0j} \tilde{B}_k$. The important simplification is that only s is present in the B_k and $\tilde{B}_k$ and in the coefficients of $\bar{n}_{03}^2$ (multiplied eventually by trivial P factors).

Let us write a, z_j, and y_3 with the s parameter:

$$\begin{gathered} a = s^{-1}(s-1)^{-1}\, 8P/(P+1)^2, \qquad y_3 = y_3^{\pm} = -(1+2/s) \pm (2/s)(s+1)^{1/2} \\ 2z_{\pm} = s'(P+1)(-1 \mp \delta^{1/2}), \qquad \delta = 1 - 4P/[s(P+1)]^2 \end{gathered} \tag{2.13}$$

If we assume $P > 0$ and, for instance, $s > 1$, then the signs of $\bar{\Sigma}_i$, $\bar{n}_{03}$, and B_i are those of Σ_i, n_{03}, and A_i. Further, a is positive, and y_3 and z_j are real. In Appendix B.4 we prove the following theorem.

Table I. $\Sigma_i = \bar{n}_{03} \bar{\Sigma}_i/(s+1)$ for the Models of Section 2

$$\begin{aligned}
\bar{\Sigma}_1^2 &= 4(\bar{n}_{03} - n_{01} B_1)(\bar{n}_{03} - n_{02} B_2) \\
\bar{\Sigma}_2^2 &= 4(\bar{n}_{03} - n_{01} B_2)(\bar{n}_{03} - n_{02} B_1) \\
\bar{\Sigma}_3^2 &= (P+1)\, P^{-1}(s-1)(\bar{n}_{03} - n_{01} B_2)(\bar{n}_{03} - n_{02} B_2) \\
\bar{\Sigma}_4^2 &= (P+1)(\bar{n}_{03} - n_{01} B_1)(\bar{n}_{03} - n_{02} B_1)\, 2s \\
\bar{\Sigma}_1^{03} &= (\bar{n}_{03} - n_{01} B_3)(\bar{n}_{03} - n_{02} B_4)\, 2(1 + y_3) \\
\bar{\Sigma}_2^{03} &= (\bar{n}_{03} - n_{01} \tilde{B}_4)(\bar{n}_{03} - n_{02} \tilde{B}_3)\, 2(1 + 1/y_3) \\
\bar{\Sigma}_3^{03} &= (P+1)\, P^{-1}(\bar{n}_{03} - n_{01} B_3)(\bar{n}_{03} - n_{01} \tilde{B}_3)(s-1) \\
\bar{\Sigma}_4^{03} &= (P+1)(\bar{n}_{03} - n_{01} \tilde{B}_4)(\bar{n}_{03} - n_{02} B_4)(-2) \\
\bar{\Sigma}_1^3 &= (\bar{n}_{03} - n_{01} B_5)(\bar{n}_{03} - n_{02} B_6)\, 2(3 + y_3) \\
\bar{\Sigma}_2^3 &= (\bar{n}_{03} - n_{01} \tilde{B}_6)(\bar{n}_{03} - n_{02} \tilde{B}_5)\, 2(3 + 1/y_3) \\
\bar{\Sigma}_3^3 &= (\bar{n}_{03} - n_{01} B_5)(\bar{n}_{03} - n_{02} \tilde{B}_5)(P+1)\, P^{-1}(s-3) \\
\bar{\Sigma}_4^3 &= (\bar{n}_{03} - n_{01} \tilde{B}_6)(\bar{n}_{03} - n_{02} B_6)(P+1)\, 2(s-1)
\end{aligned}$$

$S = -s(P+1)$, $\bar{n}_{03} = n_{03} P/(P+1)$, $y_3^{\pm} = -(1+2/s) \pm (s+1)^{1/2}(2/s)$

$B_1 = (s-1)/4$, $B_2 = s/2$, $B_3 = -(1+y_3)^{-1}$, $B_4 = (s-1)\, y_3/2(1+y_3)$

$B_5 = (s-1)/(3+y_3) = 2B_6(s-1)/(s-3)$

$B_6 = [2s + (s-1)\, y_3]/2(3+y_3) = [-2y_3 + s(1+y_3)]/4(1+y_3)$

$\tilde{B}_3 = B_3 y_3$, $\tilde{B}_4 = B_4/y_3$, $\tilde{B}_5 = (s-1)\, y_3/(3y_3+1)$

$\tilde{B}_6 = [s(1+y_3) - 2]/4(1+y_3)$

Theorem 1. The Σ_i are positive if the following sufficient conditions on the arbitrary parameters are satisfied:

$$s>3, \qquad P>0, \qquad y=y_3^-, \qquad 0<n_{01}<n_{02}B_6/B_1, \\ n_{01}B_3<n_{03}P/(P+1)<n_{01}B_1 \tag{2.14}$$

with $4B_1=s-1$, $B_3=-1/(1+y_3)$, $B_6=-[2s+(s-1)y_3]/2(3+y_3)$ positive. We explain this result. Σ_i^3 and Σ_i^2, with eight positive roots, are positive if $\bar{n}_{03}$ is smaller than their lowest root, which is $n_{01}B_1$ if $n_{01}/n_{02}<B_6/B_1<1$. With Σ_i^{03} remaining positive inside the interval $(n_{01}B_3, n_{02}B_4)$, the inequality $B_1<B_4$ leads to (2.14).

The Σ_i are really asymptotic N_i liits if $\tau_1\tau_2>0$. Since $\tau_1\tau_2$ is (Appendix B.5) the product of two quadratic $\bar{n}_{03}$ polynomials with two positive roots, it remains positive for $\bar{n}_{03}$ smaller than these roots. This is true for $\bar{n}_{03}$ in the (2.14) interval. In conclusion, Theorem 1 leads to a class of positive N_i.

What are the possible a values in (2.14)? From (2.13) we see that $a<1/3$, so that the $a=1$ value for the microreversibility is not possible.

In Section 4 we fully discuss a numerical example with a small a value and d_j parameters chosen so that $N_i>0$ in the whole x, y plane. Here, as illustration, for a solution satisfying (2.14) with $a=0.3$ we report the numerical values for both the parameters of the N_i and the Σ_i. Starting with $s=3.12$ (or $S=-6.86$), $P=1.2$, and $n_{02}=1$, we find $a=0.3$, $z_+=0.18$, $z_-=-6.68$, $y_3^-=-2.94$, $n_{01}\sup=0.042$, $0.515n_{01}<6n_{03}/11<0.53n_{01}$. Choosing further $n_{01}=32\times10^{-2}$ and $n_{03}=31\times10^{-2}$, we obtain $n_{j1}=-1.06$, -1.04, 9×10^{-5}; $n_{j2}=-1.06$, 1.04, -3×10^{-3}; $n_{j3}=0.45$, 0.24, 8×10^{-4}; $n_{j4}=-3.02$, -0.42, 10^{-4}; $\tau_j=1.95$, 1.91, 58×10^{-3}; $\rho_j=1.45$, -1.33, 0.64; $\gamma_j=0$, 0, -0.13, $j=1, 2, 3$; $\Sigma_i^2=10^{-2}$, 0.97, 2.8, 10^{-2}; $\Sigma_i^{03}=3\times10^{-2}$, 1, 3×10^{-2}, 3.3; $\Sigma_i^3=10^{-2}$, 0.98, 2.9, 10^{-2}, $i=1, 2, 3, 4$.

3. MODELS WITH TWO SIMILARITY COMPONENTS DEPENDENT SPATIALLY ON ONLY $y+\mu x$

We study the $(2+1)$-dimensional solutions

$$N_i=n_{0i}+\sum_{j=1}^{3} n_{ji}/D_j \\ D_j=1+d_j\exp(\tau_j y+\gamma_j x+\rho_j t), \qquad \gamma_j=\mu\tau_j, \qquad j=1,2 \tag{3.1}$$

The first two $j=1, 2$ components are spatially dependent on only $y+\mu x$ at $t=0$ and we recover the previous model for $\mu=0$. Our aim is still to prove analytically that there exists a class of solutions N_i such that the 16 asymptotic shock limits Σ_i defined in (2.2) are positive. We have one more parameter, μ; however, *we assume that the microreversibility* $a=1$ *is satisfied*, so that we still have five arbitrary parameters from which we deduce all others.

First we define the same five arbitrary parameters as in Section 2:

$$(z_j = n_{j4}/n_{j3},\ j=1, 2 \rightarrow P = z_1 z_2,\ S = z_1 + z_2;\ n_{0i},\ i=1, 2, 3) \qquad (3.2)$$

The connection between the first two components and the third one is still established with $y_3 = n_{31}/n_{32}$. However, y_3 is given by a cubic equation; this leads to a more complicated formalism for the analytic expression of the solutions in terms of the arbitrary parameters P and S.

Second, we write down the 16 Σ_i quantities in terms of the arbitrary parameters. Due to the y_3 cubic equation and the complication of the formalism, we must keep in the expressions intermediate parameters $\mu(P, S)$, $y_3(P, S)$, $\bar{n}_{3i}(P, S) = n_{3i}/n_{32}$, $i=1, 2$. As in Section 2, the Σ_i can be written down as linear combinations of the four n_{0i}. A remarkable property arises, which unfortunately has only been verified in each case, but has not been deduced on a fundamental basis. We find always that the coefficient of n_{04} is the product of the two corresponding ones for n_{01} and n_{02}. This allows to write $n_{03}\Sigma_i$ as a second-degree n_{03} polynomial

$$\Sigma_i = \Omega_i \left(n_{03} + \sum_{j \neq 3} \alpha_{ji} n_{0j} \right) \qquad \text{if} \quad \alpha_{4j} = \alpha_{1j}\alpha_{2j}$$

$$\rightarrow n_{03}\Sigma_i = \Omega_i (n_{03} + \alpha_{1j} n_{01})(n_{03} + \alpha_{2j} n_{03}) \qquad (3.3)$$

with Ω_i and α_{ij} only P, S dependent. Fortunately, invariance properties $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ allow us to establish this factorization property only for $i=1$ and 3 and to deduce it for $i=2$ and 4. The factorization property (3.3) simplifies the study of the positivity of the Σ_i. We look at the signs of both the coefficient of n_{03}^2 and of the roots n_{0j} multiplied by P, S functions. From this we can decouple the P, S parameters from the n_{0j} ones. The study of the intersections of the different n_{03} intervals in which the Σ_i are positive is mainly reduced to a study of P, S-dependent functions.

Third, we seek a domain of the arbitrary parameter space in which $\Sigma_i > 0$. The analytic expressions of μ, y_3, $\bar{n}_{3i}$ as functions of P, S are very complicated in general, so we choose a simplified case occurring for $S = -2(P+1)$, $0 < P < 1$.

3.1. Solutions N_i (Appendices C.1 and C.2)

There exist 20 relations among the 25 parameters n_{0i}, n_{ji}, τ_j, γ_j, ρ_j, which must be determined from the five $(P, S, n_{0i},\ i=1,2,3)$ arbitrary ones. μ is only P, S dependent, while n_{04} is only n_{0j} dependent:

$$\mu^2 = (1+P+S)^2/(1+P-S)^2 - 8P(1+P+S)/S(1+P-S)^2, \qquad n_{04} = n_{01}n_{02}/n_{03} \tag{3.4}$$

We notice that we have two square-root determinations for μ.

We discuss first the reconstruction of the $j=1,2$ components. We introduce the intermediate parameters $\bar{n}_{ji} = n_{ji}/n_{j3}$,

$$\begin{gathered} \bar{n}_{j1} = 2z_j/C_j, \qquad \bar{n}_{j2} = 2z_j/E_j \\ C_j = \mu - 1 - (\mu+1)z_j, \qquad E_j = C_j(\mu \to -\mu) \end{gathered} \tag{3.5}$$

which are functions of $\mu(P, S)$, P, and S. We obtain the n_{j3} parameter:

$$n_{j3} = (-n_{03}z_j - n_{04} + n_{01}\bar{n}_{j2} + n_{02}\bar{n}_{j1})/(z_j - \bar{n}_{j1}\bar{n}_{j2}) \tag{3.6}$$

from which we can obtain all the others, $n_{j4} = z_j n_{j3}$, $n_{ji} = \bar{n}_{ji}n_{j3}$, $i=1,2$, and τ_j, γ_j, and ρ_j [Eq. (C.4)]. For the third component $j=3$, the intermediate parameters y_3 and $\bar{n}_{3i}$ are linked by the relations

$$\begin{gathered} \bar{n}_{33}/2 = (\mu-1)/C_1C_2 - (\mu+1)\,y_3/E_1E_2 \\ \bar{n}_{34}/2P = -(\mu+1)/C_1C_2 + (1-\mu)\,y_3/E_1E_2 \\ (\bar{n}_{33} + \bar{n}_{34})\,y_3 + \bar{n}_{33}\bar{n}_{34}(1+y_3) = 0 \end{gathered} \tag{3.7}$$

with coefficients that are functions of $\mu(P, S)$, P, and S. We notice that the elimination of $\bar{n}_{3i}$ in (3.7) leads to a cubic equation for y_3, which is written down in Eq. (C.6). We find for n_{32} an expression which allows us to determine all n_{3i} as well as τ_3, γ_3, and ρ_3 [see Eqs. (C.7) and (C.8):

$$n_{32} = (n_{03}\bar{n}_{34} + n_{04}\bar{n}_{33} - n_{01} - n_{02}\,y_3)/(y_3 - \bar{n}_{33}\bar{n}_{34}) \tag{3.8}$$

3.2. Analytic Expressions of the Σ_i (Appendix C.3 and Table II)

In Appendix C the study is performed for the three families Σ_i^{03}, Σ_i^2, and Σ_i^3. Here, as illustration, we make explicit the simplest case Σ_i^{03} for which the factorization property is trivial. Further, we show briefly how the invariance properties allow one to find Σ_2 and Σ_4.

We start with $\Sigma_1^{03}=n_{01}+y_3n_{32}$, which from (3.8) can be written as a linear combination of the n_{0i}:

$$\begin{gathered}\Sigma_1^{03}=\Omega_1^{03}(n_{03}+n_{04}\bar{n}_{33}/\bar{n}_{34}-n_{02}\,y_3/\bar{n}_{34}-n_{01}\bar{n}_{33}/y_3)\\ \Omega_1^{03}=y_3\bar{n}_{34}/(y_3-\bar{n}_{33}\bar{n}_{34})\end{gathered}\tag{3.9}$$

Since the coefficient of n_{04} is the product of those of n_{01} and n_{02}, with (3.4) for n_{04} we can rewrite

$$\begin{gathered}n_{03}\Sigma_1^{03}=\Omega_1^{03}(n_{03}-n_{01}A_3)(n_{03}-n_{02}A_4),\\ A_4=y_3/\bar{n}_{34},\qquad A_3=\bar{n}_{33}/y_3\end{gathered}\tag{3.10}$$

The coefficient Ω_1^{03} of n_{03}^2 is only P, S dependent and the ratio of the two roots is n_{01}/n_{02} multiplied by A_3/A_4, still a P, S factor. For the other Σ_i^{03} this factorization structure is also trivial to establish (Section C.31); it becomes tedious [(C.32), (C.33)] for Σ_i^2 and Σ_i^3.

We report briefly the main results established in Appendix C.3. The quadratic polynomials $n_{03}\Sigma_i$ are of the type

$$n_{03}\Sigma_i=\Omega_i(n_{03}-n_{01}A_k)(n_{03}-n_{02}A_{k'})\tag{3.11}$$

with A_k, $A_{k'}$, and Ω_i functions of P and S and of the intermediate parameters y_3, $\bar{n}_{3i}$, and μ. For the transformations $1\leftrightarrow 2$ and $3\leftrightarrow 4$ *we consider the intermediate parameters as independent variables*, although they are also P, S dependent.

3.2.1. Exchange $1\leftrightarrow 2$. For this transform we must change both $n_{01}\leftrightarrow n_{02}$ and $n_{j1}\leftrightarrow n_{j2}$. Then (3.11) becomes

$$\begin{gathered}n_{03}\Sigma_1\to n_{03}\Sigma_2=\Omega_2(n_{03}-n_{01}\tilde{A}_{k'})(n_{03}-n_{02}\tilde{A}_k)\\ \Omega_2=\Omega_1(n_{j1}\leftrightarrow n_{j2}),\qquad \tilde{A}_k=A_k(n_{j1}\leftrightarrow n_{j2}),\qquad \tilde{A}_{k'}=\cdots\end{gathered}\tag{3.11$'$}$$

Of course in this transformation, the factorization remains.

First, we consider Σ_1^2 with Ω_1^2 and the roots proportional to A_1, A_2 [written down in Eq. (C.18) and in Table II]. From (3.5) the transform $n_{j1}\leftrightarrow n_{j2}$ is equivalent to $C_j\leftrightarrow E_j$ or $\mu\leftrightarrow -\mu$. Consequently, we find $\Omega_2^2=\Omega_1^2(\mu\to -\mu)$ and $\tilde{A}_i=A_i(\mu\to -\mu)$.

Second, for Σ_1^{03} written down in (3.10) we exchange $n_{31}\leftrightarrow n_{32}$ or equivalently for the intermediate parameters $y_3\to y_3^{-1}$ and $\bar{n}_{3i}/y_3\to \bar{n}_{3i}$. Consequently, for Σ_2^{03} we obtain $\Omega_2^{03}=\tilde{\Omega}_1^{03}$ and the roots $\tilde{A}_k$.

Third, for Σ_1^3, with Ω_1^3 and the roots A_5 and A_6 written down in Eq. (C.23) and in Table II for the change $\Sigma_1^3\to\Sigma_2^3$ we must perform $\mu\to -\mu$, $y_3\to y_3^{-1}$, and $\bar{n}_{3i}/y_3\to \bar{n}_{3i}$. As an illustration, let us start with the root $n_{01}A_5$ with $A_5=(\Omega_1^{03}/\bar{n}_{34}+A_1\Omega_1^2)/\Omega_1^3$, which becomes the root $n_{02}\tilde{A}_5$

of Σ_2^3. In the transformation $\Omega_1^3 = \Omega_1^2 + \Omega_1^{03}$ becomes $\Omega_2^3 = \Omega_2^2 + \Omega_2^{03}$, $A_2 \Omega_1^2 \to \tilde{A}_2 \Omega_2^2$, while $\Omega_1^{03}/\bar{n}_{34} \to \Omega_2^{03}\, y_3/\bar{n}_{34}$.

3.2.2. Exchange $3 \leftrightarrow 4$.

We must change both $n_{03} \leftrightarrow n_{04} = n_{01} n_{02}/n_{03}$ and $n_{j3} \leftrightarrow n_{j4}$. We define $(\cdot)^T = (n_{j3} \leftrightarrow n_{j4})$ and start with (3.11) for Σ_3,

$$n_{03} \Sigma_3 \to n_{03} \Sigma_4 = (\Omega_3 A_k A_{k'})^T \, [n_{03} - n_{01}/(A_{k'})^T][n_{03} - n_{02}/(A_k)^T] \qquad (3.11'')$$

and we see that the factorization property holds for Σ_4 if it exists for Σ_3.

Table II. $\Sigma_i = \bar{\Sigma}_i \Omega_i / n_{03}$ **for the Models of Section 3**

$\bar{\Sigma}_1^2 = (n_{03} - n_{01} A_1)(n_{03} - n_{02} A_2)$	$\bar{\Sigma}_2^2 = (n_{03} - n_{01} \tilde{A}_2)(n_{03} - n_{02} \tilde{A}_1)$
$\bar{\Sigma}_3^2 = (n_{03} - n_{01} \tilde{A}_2)(n_{03} - n_{02} A_2)$	$\bar{\Sigma}_4^2 = (n_{03} - n_{01} A_1)(n_{03} - n_{02} \tilde{A}_1)$
$\bar{\Sigma}_1^{03} = (n_{03} - n_{01} A_3)(n_{03} - n_{02} A_4)$	$\bar{\Sigma}_2^{03} = (n_{03} - n_{01} \tilde{A}_4)(n_{03} - n_{02} \tilde{A}_3)$
$\bar{\Sigma}_3^{03} = (n_{03} - n_{01} A_3)(n_{03} - n_{02} \tilde{A}_3)$	$\bar{\Sigma}_4^{03} = (n_{03} - n_{01} \tilde{A}_4)(n_{03} - n_{02} A_4)$
$\bar{\Sigma}_1^3 = (n_{03} - n_{01} A_5)(n_{03} - n_{02} A_6)$	$\bar{\Sigma}_2^3 = (n_{03} - n_{01} \tilde{A}_6)(n_{03} - n_{02} \tilde{A}_5)$
$\bar{\Sigma}_3^3 = (n_{03} - n_{01} A_5)(n_{03} - n_{02} \tilde{A}_5)$	$\bar{\Sigma}_4^3 = (n_{03} - n_{01} \tilde{A}_6)(n_{03} - n_{02} A_6)$

General case Σ_i^2: $A_1 = -\lambda_{11}/\lambda_{31}$, $A_2 = -\lambda_{21}/\lambda_{31}$

$\tilde{A}_k = A_k(\mu \to -\mu)$, $\lambda_{11} = 4P(1+P-S)/S$

$\lambda_{21} = 2(4P - S^2) E_1 E_2 / C_1 C_2$

$\lambda_{31} = [\mu(P+1-S) + P + S + 1](S - 2P) + 4P(P-1)$

$C_1 C_2 = E_1 E_2(\mu \to -\mu)$

$E_1 E_2 = 2(1+P+S)[S(P+1) - 4P]/S(1+P-S) - 2\mu(P-1)$

$\bar{\Omega}_i^2 = \Omega_i^2 (1-\mu^2)(1+P-S)^2/2$, $\bar{\Omega}_1^2 = \lambda_{31}$

$\Omega_2^2 = \Omega_1^2(\mu \to -\mu)$, $\bar{\Omega}_3^2 = -\lambda_{11}$, $\bar{\Omega}_4^2 = 2P(S^2 - 4P)$

Σ_i^{03}: $y_3 A_3 = \bar{n}_{33} = \tilde{A}_3$, $A_4/y_3 = 1/\bar{n}_{34} = \tilde{A}_4$

$\Omega_2^{03} = \bar{n}_{34}/(y_3 - \bar{n}_{33}\bar{n}_{34}) = \Omega_1^{03}/y_3 = \Omega_4^{03}/\bar{n}_{34} = \bar{n}_{34}\Omega_3^{03}/y_3$

Σ_i^3: $A_5 = (\Omega_3^{03} + A_1 \Omega_1^2)/\Omega_1^3$, $A_6 = (\Omega_3^{03}\, y_3 + A_2 \Omega_1^2)/\Omega_1^3$

$\tilde{A}_5 = (\Omega_3^{03} + \tilde{A}_1 \Omega_2^2)/\Omega_2^3$, $\tilde{A}_6 = (\Omega_4^{03} + \tilde{A}_2 \Omega_2^2)/\Omega_2^3$

$\Omega_i^3 = \Omega_i^2 + \Omega_i^{03}$ $(i = 1, 2, 4)$, $\Omega_3^3 = \bar{n}_{33} \Omega_2^{03} + \Omega_3^2$

$\mu^2 = (1+P+S)^2/(1+P-S)^2 - 8P(1+P+S)/S(1+P-S)^2$

Particular case $S = -2(P+1)$, $\mu = (1-P)/3(1+P)$

Σ_i^2: $A_1 = 1/2P$, $A_2 = 2/P$, $\tilde{A}_2 = 2/Q = 1/\tilde{A}_1$

$\Omega_1^2 = 6P^2/(P+2)(2P+1) = 2P\Omega_3^2$

$\Omega_2^2 = 2(1+P+P^2)/(P+2)(2P+1) = \Omega_4^2/2P$

Σ_i^{03}: idem

Σ_i^3: $A_5 = (Q^2 + y_3^2 + Qy_3 - 2y_3)/(2Q + y_3)(PQ - y_3)$

$A_6 = (2Q + y_3)/(PQ - y_3)$, $\tilde{A}_6 = (2y_3 + Q)/Q(y_3 - PQ)$, $\tilde{A}_5 = A_5 A_6/\tilde{A}_6$

$\bar{\Omega}_i^3 = \Omega_i^3 (P+2)(2P+1)(Q^2 + Qy_3 + y_3^2)/3P(QP - y_3)$

$\bar{\Omega}_1^3 = 2Q + y_3$, $\bar{\Omega}_2^3 = -2y_3 - Q$

Ω_i^3, $i = 3, 4$: idem

First, for Σ_i^{03} we have $\bar{n}_{33} \leftrightarrow \bar{n}_{34}$; the roots $n_{01}A_3$ and $n_{02}\tilde{A}_3$ become the roots $n_{02}/(A_3)^T = n_{02}A_4$ and $n_{01}/(\tilde{A}_3)^T = n_{01}\tilde{A}_4$ of Σ_4^{03} [see (C.11)]. Similarly, $(\Omega_3^{03}A_3A_4)^T = \Omega_3^{03}\bar{n}_{34}^2/\bar{n}_{33} = \Omega_4^{03}$.

Second, for Σ_3^2 we have $P \to 1/P$, $S \to S/P$, and $\bar{n}_{j1} = n_{j1}/n_{j3} \to n_{j1}/n_{j4} = \bar{n}_{j1}/z_j$ or, from (3.5), $z_j \to 1/z_j$, $\mu \to -\mu$.

Consequently, for $\Sigma_3^2 \to \Sigma_4^2$ in (3.11″) the transform $(\cdot)^T$ is $(P \to 1/P$, $S \to S/P$, $\mu \to -\mu)$, leading to $1/(A_2)^T = A_1$ and $1/(\tilde{A}_2)^T = \tilde{A}_1$ and the two roots $n_{01}\tilde{A}_2$ and $n_{02}A_2$ of Σ_3^2 become the roots $n_{02}\tilde{A}_1$ and $n_{01}A_1$ of Σ_3^2. Finally, the coefficient Ω_3^2 of Σ_3^2 is transformed into $(\Omega_3^2A_2\tilde{A}_2)^T = \Omega_3^2/A_1\tilde{A}_1 = \Omega_4^2$ for Σ_4^2.

Third, for the transform $\Sigma_3^3 \to \Sigma_4^3$ we must consider $(\bar{n}_{33} \leftrightarrow \bar{n}_{34}$, $P \to 1/P$, $S \to S/P$, $\mu \to -\mu)$. Similarly as above (with tedious calculations), one can show that the two roots $n_{01}A_5$ and $n_{02}\tilde{A}_5$ of Σ_3^3 become the two roots $n_{02}/(A_5)^T = n_{02}A_6$ and $n_{01}/(\tilde{A}_5)^T = n_{01}\tilde{A}_6$ of Σ_4^3.

In conclusion, for the 12 Σ_i we only have 12 roots: $n_{01}A_{2k+1}$, $n_{02}A_{2k}$, $n_{02}\tilde{A}_{2k+1}$, $n_{01}\tilde{A}_{2k}$, $k = 0, 1, 2$; six of them are obtained by transformations of the other six.

We restrict the study to a simplified case: $S = -2(P+1)$ and $\mu = (1-P)/3(1+P)$ for the square-root determination of μ^2.

3.3. Sufficient Conditions for $\Sigma_i > 0$ and $S = -2(P+1)$, $3\mu = (1-P)/(1+P)$ (Appendix C.4)

The analytic expressions of the 12 Ω_i and of the 12 roots are written down in Table II as functions of P and of the intermediate parameters and μ, which are in fact also only P dependent. Further, in order to be able to discuss analytically the determination of the cubic y_3 equation, it is convenient to introduce another parameter Q which is also P dependent:

$$\begin{gathered} Q = 3P/(1+P+P^2), \qquad y_3^3 + Q^2 + y_3(y_3+Q)(4+Q) = 0 \\ 3P\bar{n}_{33} = Q(2P+1) + y_3(P+2), \qquad 3\bar{n}_{34} = Q(P+2) + y_3(2P+1) \end{gathered} \tag{3.12}$$

We choose the following determinations of P, Q, and y_3:

$$0 < P < 1, \qquad 0 < Q < 1, \qquad -1 < y_3 < 0 \tag{3.13}$$

The interest of the introduction of the parameter $Q(P)$ is that we can replace the cubic y_3 equation (not convenient for an analytic discussion of its solution as function of a parameter) by two quadratic ones $P(Q)$ and $Q(y_3)$ in (3.12), choosing the square-root determinations compatible with (3.13):

$$\begin{gathered} P\beta = 1 - (1-\beta^2)^{1/2}, \qquad \beta = 2Q/(3-Q) \\ Q\alpha = -2y_3[1-(1-\alpha)^{1/2}], \qquad \alpha = 4(1+y_3)/(4+y_3) \end{gathered} \tag{3.14}$$

For the positivity study we will have to compare the different roots proportional to $A_k, \tilde{A}_k$. Then the representations (3.14) give useful information about the signs of algebraic expressions involving the intermediate parameters, P and Q, while $z_j = z_{\pm} = -P-1 \pm (1+P+P^2)^{1/2}$.

Starting with (3.13) for Q, P, y_3, and $n_{0i} > 0$, $i=1, 2, 3$, we find $n_{04} > 0$ and two subdomains of (3.2) in which the Σ_i are positive. All Ω_i^2, Ω_i^3, A_k, $\tilde{A}_k$, $k=1, 2, 5, 6$, are positive, so it follows that Σ_i^2 and Σ_i^3 are positive for n_{03} outside the eight intervals constitued by the roots. We can choose either n_{03} smaller than the smallest root or larger than the largest one. Further, the roots proportional to n_{01} or n_{02} can be ordered if the ratio n_{01}/n_{02} has a P-dependent lower or upper bound. For Σ_i^{03} the two roots proportional to A_3, $\tilde{A}_4$, and Ω_i^{03}, $i=1, 3$, are positive, while $\tilde{A}_3$, A_4, and Ω_i^{03}, $i=2, 4$, are negative. Only for $A_3 < n_{03}/n_{01} < \tilde{A}_4$ can the positivity be satisfied. Choosing for Σ_i^2 and Σ_i^3 the n_{03} interval smaller (larger) than the smallest (largest) root, then this root must be larger (smaller) than $n_{01}A_3$ ($n_{01}\tilde{A}_4$). An analytic positivity proof requires a great deal of algebraic calculation. We must order the A_k, $\tilde{A}_k$ (20 lemmas) and intermediate results are found: $\bar{n}_{33} < 0$, $\bar{n}_{34} > 0$, $\bar{n}_{33}\bar{n}_{34} - y_3 > 0$, $y_3 + Q > 0$, $2y_3 + Q < 0$.... We have proved the following theorem[3] for n_{03} smaller (larger) than the Σ_i^2, Σ_i^3 roots.

Theorem 2. Sufficient conditions for all $\Sigma_i > 0$ are

$$0 < P < 1 \rightarrow (-1 < y_3 < 0), \qquad 0 < n_{01}/n_{02} < PQ, \qquad A_3 < n_{03}/n_{01} < \tilde{A}_6$$

$$A_3 = \bar{n}_{33}/y_3, \qquad \tilde{A}_6 = (2y_3 + Q)/Q(y_3 - PQ), \qquad Q(1+P+P^2) = 3P$$

Theorem 3. Sufficient conditions for all $\Sigma_i > 0$ are

$$0 < P < 1, \quad (-1 < y_3 < 0), \qquad n_{01}/n_{02} > Q/P, \quad \sup(\tilde{A}_2, A_5) < n_{03}/n_{01} < \tilde{A}_4$$

$$\tilde{A}_2 = 2/Q, \quad A_5 = (Q^2 + y_3^2 + Qy_3 - 2y_3)/(2Q + y_3)(QP - y_3), \quad A_4 = 1/\bar{n}_{34}$$

It is shown in Appendix C.4 that $\tau_1\tau_2$, having two positive roots, remains positive for n_{03} outside the interval constituted by these roots. This is the case for the n_{03} intervals of Theorems 2 and 3. These theorems lead to positive Σ_i and N_i.

In Section 4 we discuss a numerical example of Theorem 2, while here we present the N_i parameers and A_k, $\tilde{A}_k$, and Σ_i numerical values for an example of Theorem 3.

First, we start with $P = 3/5$, leading to $S = -3.2$, $z_+ = -0.2$, $z_- = -3$; $Q = 45/49$, $y_3 = -0.88$, $\bar{n}_{33} = -1.5$, $\bar{n}_{34} = 0.15$, $\mu = 1/12$; $A_1 = 5/6$, $\tilde{A}_1 = 45/98$, $A_2 = 10/3$, $\tilde{A}_2 = 98/45$, $A_3 = 0.17$, $A_4 = -5.9$, $\tilde{A}_3 = -0.15$, $\tilde{A}_4 = 6.71$, $A_5 = 2.53$, $A_6 = 0.666$, $\tilde{A}_5 = 2.62$, $\tilde{A}_6 = 0.643$; $\Omega_i^3 = 0.5$, 0.5, 0.3, 0.8, $\Omega_i^{03} = 1.5$, -0.17, 1, -0.02, $i = 1, 2, 3, 4$; $n_{01}/n_{02} > 75/49$, $2.53 < n_{03}/n_{01} > 6.71$.

Second, we choose $n_{02}=1$, $n_{01}=2.224$, and find $5.62<n_{03}<14.9$.

Third, we choose $n_{03}=8.73$ and find $n_{04}=0.25$, $n_{03,z_+}=1.23$, $n_{03,z_-}=3.7$; $n_{j1}=-3.84,\ 3.22,\ -0.08$, $n_{j2}=-2.97,\ 4.5,\ 0.09$, $n_{j3}=-6.72,\ -1.25,\ -0.014$, $n_{j4}=1.34,\ 3.75,\ 0.014$, $\tau_j=-9.1,\ -10.2,\ 0.54$, $\gamma_j=-0.76,\ -0.85,\ -0.08$, $\rho_j=6.1,\ -5.1,\ -0.054$, $j=1, 2, 3$; $\Sigma_i^2=1.6,\ 2.5,\ 0.7,\ 5.3$, $\Sigma_i^{03}=2.1,\ 1.1,\ 8.7,\ 0.27$, $\Sigma_i^3=1.5,\ 2.6,\ 0.74,\ 5.4$, $i=1, 2, 3, 4$.

4. PHYSICAL DISCUSSION

We consider the square-velocity model (the discussion is similar for the cubic one) with the total mass $M=\sum_1^4 N_i$ rewritten as

$$\begin{gathered}M=m_0+\sum_1^3 m_j/D_j, \qquad m_0=\sum_1^4 n_{0i}, \qquad m_j=\sum_1^4 m_{ji}\\ D_j=1+d_j\exp(\tau_j\bar{y}+\rho_j t), \qquad j=1,2\\ D_3=1+d_3\exp(\tau_3\bar{y}+\bar{\gamma}_3 x+\rho_3 t)\end{gathered} \tag{4.1}$$

We have introduced a new coordinate $\bar{y}=y+\mu x$ for the models of Section 3, while for the spatial coordinates in D_3 we put $\tau_3 y+\gamma_3 x=\tau_3\bar{y}+\bar{\gamma}_3 x$ ($\bar{\gamma}_3=\gamma_3-\mu\tau_3$). We discuss the solutions with $\bar{y}$ and x as spatial coordinates. For the model of Section 2, $\mu=0$, $\bar{y}=y$, $\bar{\gamma}_3=\gamma_3$.

The previous positivity conditions $\Sigma_i>0$ for N_i become for M

$$m_0>0, \qquad \Sigma^2=\sum_0^2 m_i>0, \qquad \Sigma^{03}=m_0+m_3>0, \qquad \Sigma^3=\sum_0^3 m_i>0 \tag{4.2}$$

if $\tau_1\tau_2>0$. We notice that (4.2) satisfied alone are insufficient conditions for $\Sigma_i>0$.

4.1. Some General Results

We first discuss the equidensity lines $M=\text{const}$ at $t_0=0$ and next the movement of the shock front and the relaxation toward equilibrium.

4.1.1. Equidensity Lines $M(x, \bar{y}, t=0)=\text{const}$. We look, in the x, $\bar{y}$ plane, at the asymptotic domains associated with the limiting M values. Depending upon whether $\tau_1\bar{y}$ (we recall $\tau_1\tau_2>0$) and $\tau_3\bar{y}+\bar{\gamma}_3 x$ are positive or negative, we find the four asymptotic shock limits of (4.2): (i) $\tau_1\bar{y}>0$, $\tau_3\bar{y}+\bar{\gamma}_3 x>0$, $M\to m_0$, (ii) $\tau_1\bar{y}>0$, $\tau_3\bar{y}+\bar{\gamma}_3 x<0$, $M\to\Sigma^{03}$, (iii) $\tau_1\bar{y}<0$, $\tau_3\bar{y}+\bar{\gamma}_3 x>0$, $M\to\Sigma^2$, (iv) $\tau_1\bar{y}<0$, $\tau_3\bar{y}+\bar{\gamma}_3 x<0$, $M\to\Sigma^3$. They define domains limited by equidensity lines parallel to $y=0$ and $\tau_3\bar{y}+\bar{\gamma}_3 x=0$. There exist four different possibilities (see Fig. 1a),

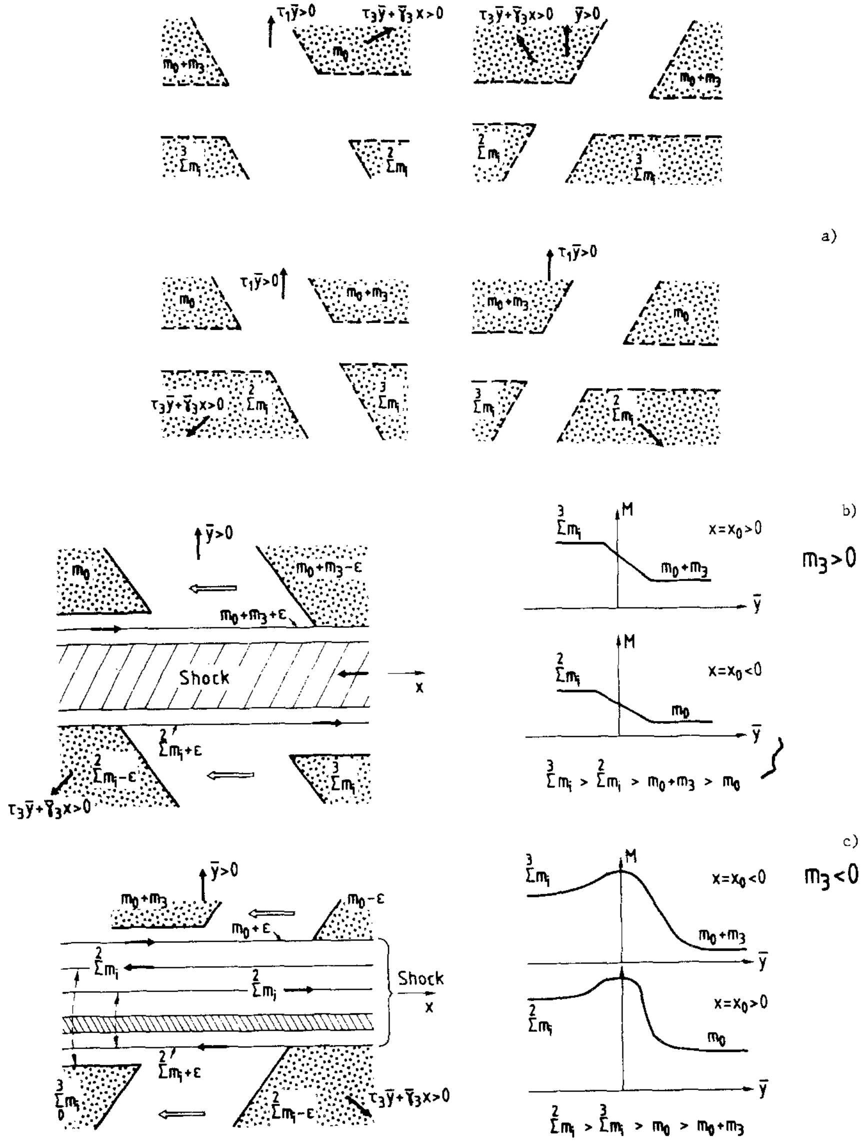


Fig. 1. (a) Different locations of the shock plateaus. (b) The shock front decreases continuously. (c) The shock front has a bump.

depending upon whether the positive $\tau_3 \bar{y} + \bar{\gamma}_3 x$ axis is in the first, second, third, or fourth quadrant of the x, $\bar{y}$ plane. In the case of one spatial coordinate, we only have two shock limits: one in the upstream domain and the other in the downstream domain. Here we can have, for instance, two asymptotic shock plateaus in the upstream domain and two others in the downstream domain. The definitions (4.2) are insufficient to order these limits and determine which ones are in the upstream or in the downstream domain.

We consider a shock in a strip parallel to the x axis and look at the possible ways for the equidensity lines to link the asymptotic plateaus of both up- and downstream domains. We will say that the upstream domain contains the two highest plateaus, while the downstream domain contains the two lowest. We are interested in the possibility that the domain around the shock has bumps higher than the highest asymptotic plateau. If $m_i > 0$ for all i, then Sup M in the whole $\bar{y}$, x plane is the highest plateau Σ^3. If some m_i is negative, then the arbitrary constants d_j in D_j [not present in (4.2)] are important. Among the possible scenarios, we choose two, which will be illustrated numerically later. We choose two opposite situations, with such bumps never or always present.

For the first scenario we assume

$$m_1 + m_2 > m_3 > 0 \rightarrow m_0 < \Sigma^{03} < \Sigma^2 < \Sigma^3 \tag{4.3}$$

and the $\tau_3 \bar{y} + \bar{\gamma}_3 x > 0$ axis in the third x, $\bar{y}$ quadrant. We choose the d_j such that in the upstream the lowest plateau Σ^2 surrounds entirely the highest one Σ^3. In Fig. 1b we represent the path for decreasing equidensity lines. Two profiles at $x = x_0$ fixed (negative and positive) show that the shock front decreases continuously from one upstream plateau to another downstream one. No bump is present. However, (4.3) is compatible with opposite signs for m_1 and m_2, for instance, $m_1 > 0$, $m_2 < 0$. Choosing d_2 large and d_1 small, we can obtain equidensity lines with M larger than Σ^3, so that bumps can appear. For instance, we can change the initial time $t_0 = 0$ to $t_0 \neq 0$ and substitute $d_2 \exp(t_0 \rho_2)$ instead of d_2 (this possibility will be illustrated later in Fig. 2).

For the second scenario we assume in Fig. 1c

$$m_1 + m_2 + m_3 > 0, \qquad m_3 < 0 \rightarrow \Sigma^{03} < m_0 < \Sigma^3 < \Sigma^2 \tag{4.4}$$

The $\tau_3 \bar{y} + \bar{\gamma}_3 x > 0$ axis in the fourth quadrant and the d_j are chosen such that in the upstream domain the lowest plateau is an hollow entirely surrounded by the highest one. It is an isolated basin from which, following decreasing equidensity lines, we cannot go directly to the shock front. Further, there is no path connecting directly up- and downstreams plateaus through the shock front. In a strip parallel to the x axis, including the shock, $\sup M(x, \bar{y})$ for x fixed and $\bar{y}$ varying is larger than the highest plateau Σ^2. A bump is always present, close to the shock front with equidensity lines higher than Σ^2, as is illustrated with two profiles at x_0 fixed positive and negative. Other scenarios are possible; for instance, m_1 and m_2 can be of opposite sign in (4.4) $(m_1+m_2>0)$ and we can choose d_1 and d_2 such that in some intervals, $\sup M$ for x fixed is lower than Σ^2.

4.1.2. Movement of the Shock Front and Relaxation toward Equilibrium. With the initial time t_0 arbitrary and without significance, we are interested in large t values and finite spatial coordinates $x, \bar{y}$ values. Among different possibilities, let us choose $\rho_3>0$ and $\rho_i>0$ for one of the two $i=1, 2$ values, while $\rho_j<0$ for the other $j\neq i$. In a first crude approximation for large time we find

$$M \simeq m_0 + m_j/[1+d_j \exp(\tau_j \bar{y} - |\rho_j| t)] \tag{4.5}$$

The shock front for large time has moved from $\bar{y}\simeq 0$ to $\bar{y}\simeq t|\rho_j|/\tau_j$. There remain practically two asymptotic plateaus m_0, m_0+m_j, the last one becoming the Maxwellian equilibrium state. We remark that $m_0+m_j>0$ is not a consequence of the positivity conditions (4.2) at $t=0$. With the Boltzmann equation carrying through the positivity, this means that for the present situations (see examples in Figs. 2 and 3), necessarily $m_0+m_j>0$.

These results represent the dominant effects, but less important ones occur. First, what happens for the equidensity lines $\tau_3\bar{y}+\bar{\gamma}_3 x=\text{const}$ (present in the asymptotic plateaus)? From D_3 we see that they are translated to $-\rho_3 t$. From the different signs of the ρ_i, we see that the plateaus Σ^3, Σ^2, Σ^{03} move toward the equilibrium state m_0+m_j and m_0. At intermediate times the ith component, proportional to m_i, gives a contribution to the shock front for movement toward $\bar{y}=-t\rho_i/\tau_1$ in a direction opposite to the dominant jth component, proportional to m_j.

4.2. An Explicit Example with $a\neq 1$

We discuss an example of the formalism of Section 2 with $D_j=1+d_j \exp(\tau_j y+\rho_j t)$, $j=1, 2$, or $\bar{y}=y$, $\bar{\gamma}_3=\gamma_3$. We choose arbitrary parameters satisfying Theorem 1: We start with $P=0.5$, $S=-15$, $n_{01}=10^{-3}$, $n_{02}=1$, $n_{03}=6.7\times 10^{-3}$, leading to $a=19.7\times 10^{-3}$, $z_1=-33\times 10^{-3}$, $z_2=-14.9$, $y_3=-1.8$, $n_{04}=7.5$; $\tau_j=1.22$, 1.22, -0.054, $\gamma_j=0$, 0, -0.54, $\rho_j=1.07$, -1.14, 0.16, $n_{j1}=-1$, 1, 0.74, $n_{j2}=-1$, 1, -0.4, $n_{j3}=0.47$, 14.5, 2.6, $n_{j4}=-7$, -4.85, 1.3, $j=1, 2, 3$. We notice that the sound speed of the two first components $y+c_j t$ and of the third one $x+y\tau_3/\gamma_3+c_3 t$ are such that $|c_i|<1$. For the total mass M we deduce $m_0=8.5$, $m_1=-8.55$, $m_2=16.04$,

$m_3 = 4.19$; $\Sigma^3 = 20$, $\Sigma^2 = 16$, $\Sigma^{03} = 12.7$. For the arbitrary d_j parameters we choose $d_1 = d_2 = 10$, $d_3 = 10^{-2}$.

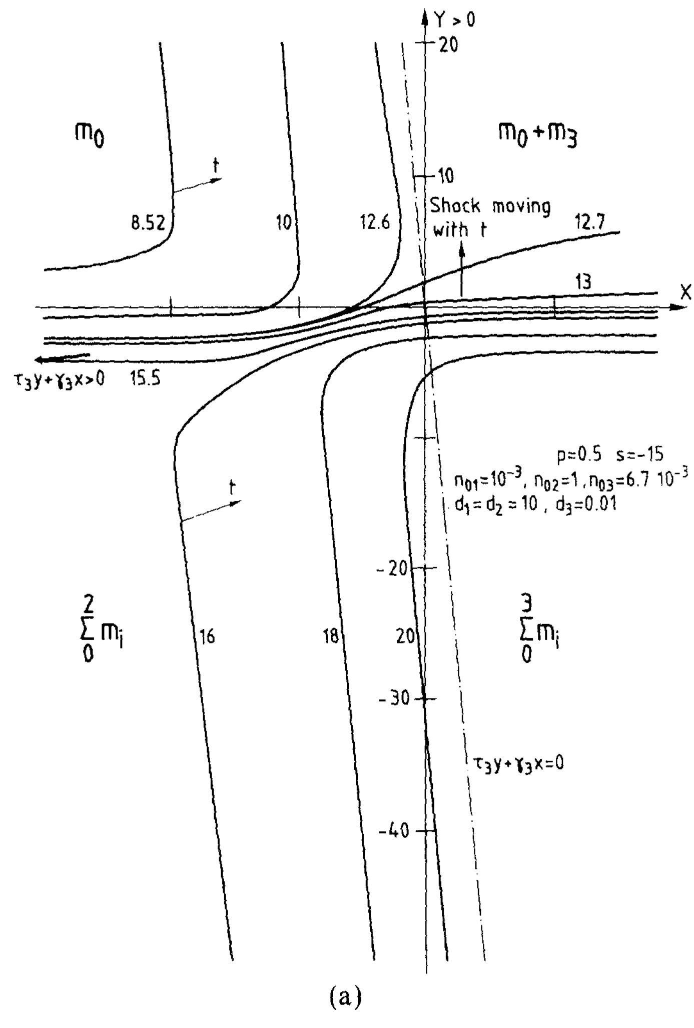
4.2.1. Equidensity Lines $M = \text{const}$ at $t = 0$ (Fig. 2a). These correspond to the scenario of Fig. 1b with a shock in a strip around the x axis. Decreasing equidensity lines connect he asymptotic plateaus $20 \to 16$, and then they cross the shock front and spread out in the downstream toward 12.7 and finally 8.5. The profiles perpendicular to the x axis decrease continuously from the upstream toward the downstream domain. We observe the equidensity lines parallel to $\tau_3 y + \gamma_3 x = 0$.

4.2.2. Shock and Equidensity Lines Moving with t, Equilibrium State (Fig. 2b–2e). For large t and finite x, y only the second component remains: $M \simeq 8.5 + 16/[1 + d_2 \exp(y - t)]$; the shock is shifted from $y = 0$ to $y = t$, relaxing toward the equilibrium state $m_0 + m_2 \simeq 24.5$, while for $y - t$ positive and large, $M \simeq m_0 = 8.5$. From the expression of D_3 we observe that the equidensity lines parallel to $\gamma_3 x + \tau_3 y = \text{const}$ are translated to $\text{const} - 0.16t$. Consequently, both plateaus 20 and 12.7 join the others, 16 and 8.5.

Looking at the d_j values for $t = t_0$ large but fixed, we observe that $d_1 \exp(t_0)$ is large compared to $d_2 \exp(-t_0)$, $d_3 \exp(0.16 t_0)$. Consequently, the negative term, proportional to m_1 becomes less important and we can observe a bump higher than 20 or even higher that the equilibrium value 24.5. This means that in the space, populations of particles larger than at the initial time or at infinite time can appear.

Figures 2b and 2c present results for N_i and M for a small y interval around the shock, along the lines $x = 10$ and $x = -y - 10$. We observe both the displacement of the shock front and, at intermediate time, the presence of a bump larger than the equilibrium state. A plot of M for some x, y fixed and t varying emphasizes the presence of this bump at intermediate times. We also notice the property, sometimes overlooked, that the positivity of the macroscopic total mass M is not sufficient to ensure the positivity of the N_i. For a small negative time $t = -1.25$, both N_i and M remain positive, but, for instance, for $t = -2$, M is still positive, while N_2 becomes negative around $y = 0$.

In order to follow the displacement of the asymptotic plateaus, Fig. 2d–2f show results for M with a large y interval. We choose the constant line $x = 10$ and the two others $\tau_3 y + \gamma_3(x \pm 10) = 0$, which are parallel to one of the two directions of the asymptotic plateaus. In Fig. 2d for $t = 20$ we observe both that the central plateau with hedge 20 becomes thinner with an enhancement at 28 and of course the moving of the shock. The compression of the central plateau can explain physically the appearance of the bump, while mathematically, as we have seen this is due to



(a)

Fig. 2. (a) The M equidensity lines at $t=0$ decrease continuously from the highest plateau to the smallest one. Lines parallel to $x=0$ and to $\tau_3 y+\gamma_3 x=0$ ($a=19.7\times 10^{-3}$). (b) A bump appears at intermediate times. Movement of the shock front ($x=10$, $a=19.7\times 10^{-3}$). (c) At $t=-2$, M is still positive, but N_2 is negative ($x+y+10=0$). (d) All the asymptotic plateaus as well as the bump and the equilibrium state are present. The bump appears and disappears ($a=19.7\times 10^{-3}$, $x=0$). (e, f) The bump is present. Pictures similar to a shock with one special coordinate. (e) $\tau_3 y+\gamma_3(x+10)=0$, (f) $\tau_3 y+\gamma_3(x-10)=0$.

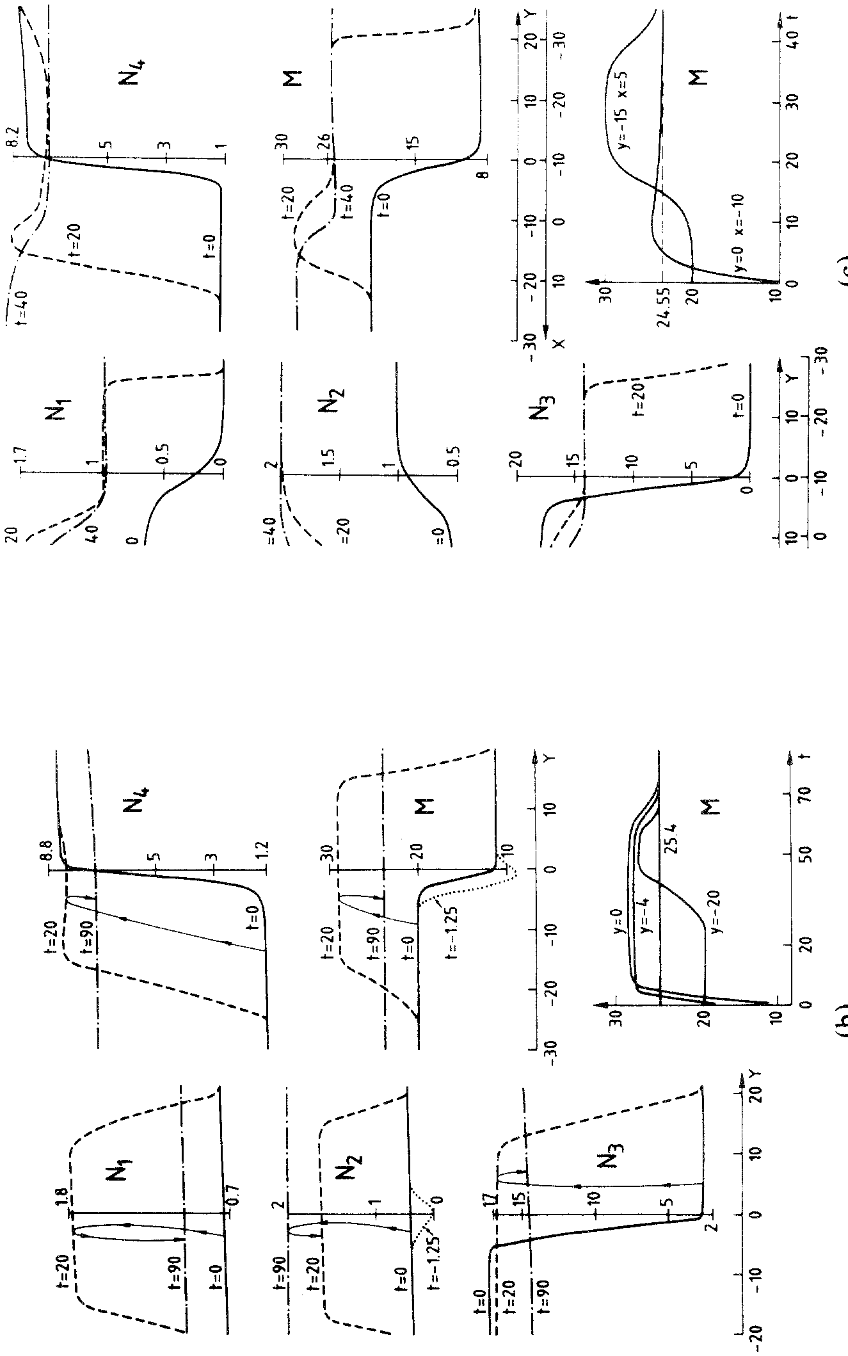


Fig. 2 (*continued*)

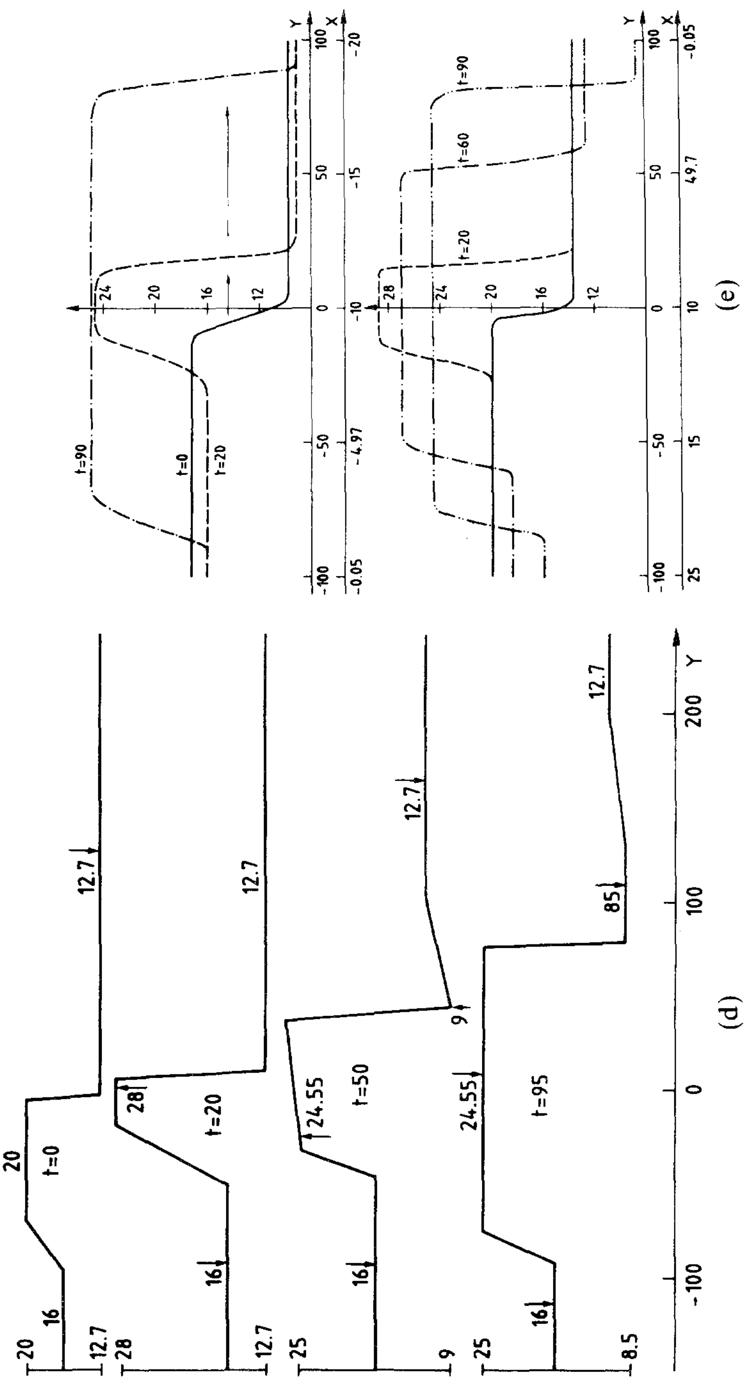


Fig. 2 (continued)

$d_1 \to d_1 \exp(20\rho_1)$. At $t=50$ the bump has disappeared and we observe the formation of the Maxwellian plateau 24.5. At $t=90$, we see four asymptotic plateaus; the smallest, 8.5, appears due to the displacement of the $\tau_3 y+\gamma_3 x=\text{const}$ line (discussion above). In Figs. 2e and 2f the two lines are parallel to $\gamma_3 x+\rho_3 t=\text{const}$, so that, as in a one-dimensional shock, we observe only two asymptotic shock limits. However, here both constant limits vary with t. We notice that the bump appears on both lines; however, on one line it is higher than the Maxwellian, but not on the other.

4.3. An Explicit Example with $a=1$

We discuss an explicit example for the formalism of Section 3 for which the $D_j=1+d_j\exp(\tau_j y+\gamma_j x+\rho_j t)$ are rewritten:

$$D_j=1+d_j\exp(\tau_j\bar{y}+\rho_j t), \qquad j=1,2$$

$$D_3=1+d_3\exp(\tau_3\bar{y}+\bar{\gamma}_3 x+\rho_3 t)$$

with $\bar{\gamma}_3=\gamma_3-\mu\tau_3$ and $\bar{y}=y+\mu x$ as a new spatial coordinate. We discuss an example with the arbitrary parameters satisfying Theorem 2.

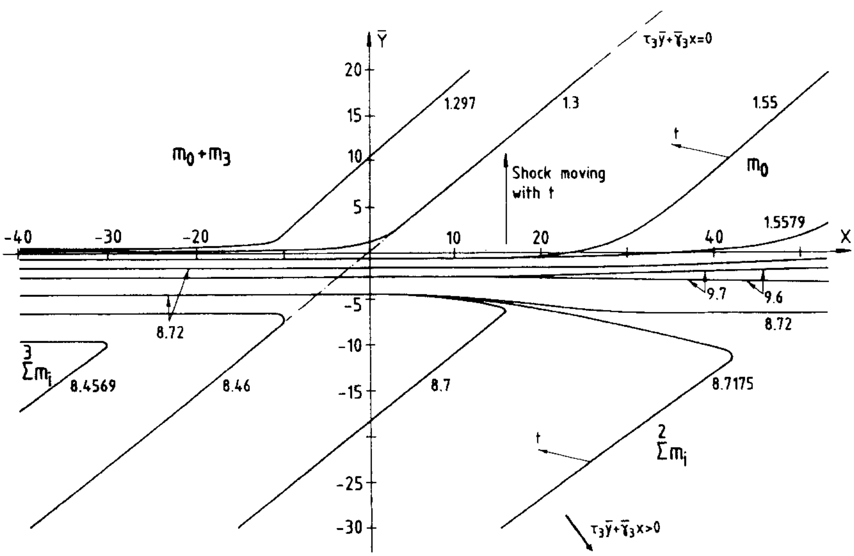
We start with $a=1$, $P=0.1$, $S=-2.2$, $n_{01}=10^{-9}$, $n_{02}=1$, $n_{03}=1.23\times 10^{-9}$, leading to $\mu=9/33$, $Q=10/37$, $z_1=-4.64\times 10^{-2}$, $z_2=-2.15$, $y_3=-0.187$, $n_{04}=0.56$; $\tau_j=4.72,\ 1.97,\ -0.393$, $\gamma_j=\mu\tau_1,\ \mu\tau_2,\ 0.19$, $\rho_j=-4.3,\ 0.72,\ 0.13$, $n_{j1}=1.02,\ -0.17,\ 0.07$, $n_{j2}=1.02,\ -0.17,\ -0.37$, $n_{j3}=7.37,\ 0.08,\ 0.85$, $n_{j4}=-0.34,\ -0.17,\ -0.042$, $j=1,2,3$. We notice that the sound speed of the first two components $y+\mu x+c_j t$ and that of the third one $x+\tau_3/\gamma_3\, y+c_3 t$ are such that $|c_i|<1$. For the total mass M we deduce $m_0=1.558$, $m_1=8.6$, $m_2=-1.44$, $m_3=-0.26$, $\Sigma^2=8.717$, $\Sigma^{03}=1.297$, $\Sigma^3=8.456$, while for arbitrary d_j we choose $d_1=d_2=10^3$, $d_3=10^{-2}$.

4.3.1. Equidensity Lines $M(\bar{y}, x, t=0)=\text{const}$ (Fig. 3a). These correspond to the scenario presented in Fig. 1c with a shock in a strip near the x axis. For the present choice of the d_j, inside the shock domain, $\sup M$ for x fixed and $\bar{y}$ varying occurs at $\bar{y}\simeq -2.5$ and varies slowly from 9.66 at $x\to-\infty$ to 9.92 when $x\to\infty$. The ridge stays practically always at 9.66 for $x<0$, rising slowly and continuously when $x>0$. All along the shock front a bump exists which isolates both the basin Σ^3 and the highest upstream asymptotic plateau Σ^2. The profiles perpendicular to the x axis exhibit this bump.

In order to test the importance of the d_i in the shock front we seek the largest and the smallest possible bumps. Due to $m_1>0$, $m_2<0$, the most important one is obtained with d_2 large and d_1 small. For $d_3=10^6$,

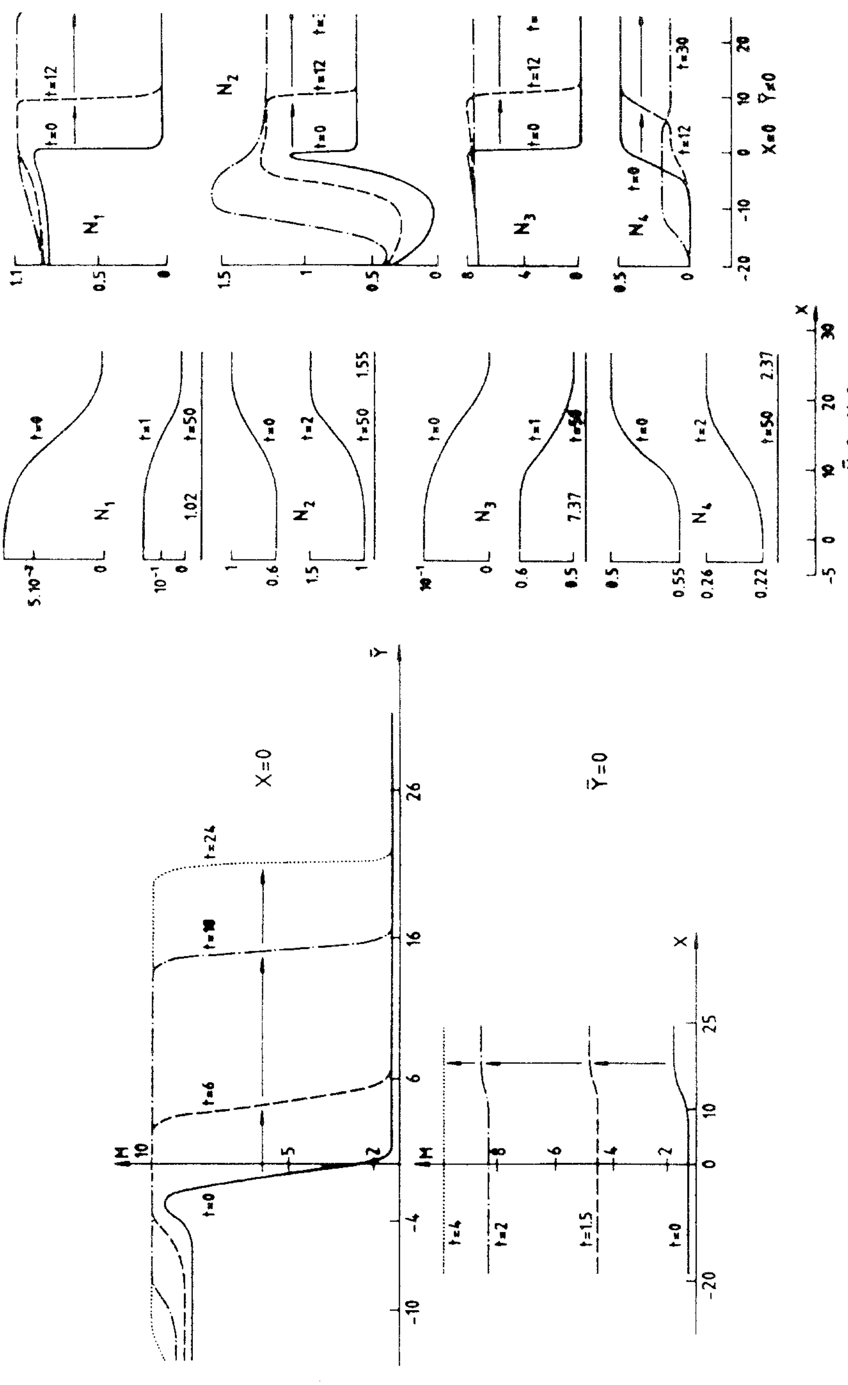
$d_1 = d_2 = 10^{-5}$ we find that sup M in the shock-strip lies between 9.9 and 10.16 (for any d_j values the difference is equal to m_3). On the contrary for $d_2 = d_3 = 10^{-4}$ and $d_1 = 10^4$, we find a sup M in the strip between $\Sigma^3 + \varepsilon$, $\Sigma^2 + \varepsilon$ $\varepsilon \simeq 10^{-5}$ with values close to the asymptotic upstream plateaus. In that equidensity lines can cross the shock domain and the bump practically disappears.

4.3.2. Shock and Equidensity Lines Moving with *t*, Equilibrium State (Fig. 3b–3d). Due to $\rho_2 > 0$, $\rho_3 > 0$, $\rho_1 < 0$, for large t and finite x, y only the first component remains: $M \simeq 1.6 + 8.6/\{1 + d_1 \exp[4.7(\bar{y} - t]\}$. The shock is moving from $\bar{y} = 0$ to $\bar{y} = t$; the equilibrium state ($t \to \infty$) has the value $m_0 + m_1 = 9.02$, while when $\bar{y} - t$ is large and positive we must observe the plateau 16. These are the dominant effects. However, for not too large time, the second component $\sim -1.45/\{1 + d_2 \exp[0.7(2\bar{y} + t)]\}$ moves in the opposite direction. The third component determines the displacement of the lines parallel to $\tau_3 \bar{y} + \bar{\gamma}_3 x = \text{const}$, which are translated to const $-1.3t$. In both upstream and downstream domains we must observe the displacement of the plateaus $\Sigma^3 \to \Sigma^2$ and $m_0 + m_3 \to m_0$. Finally, we notice that the change



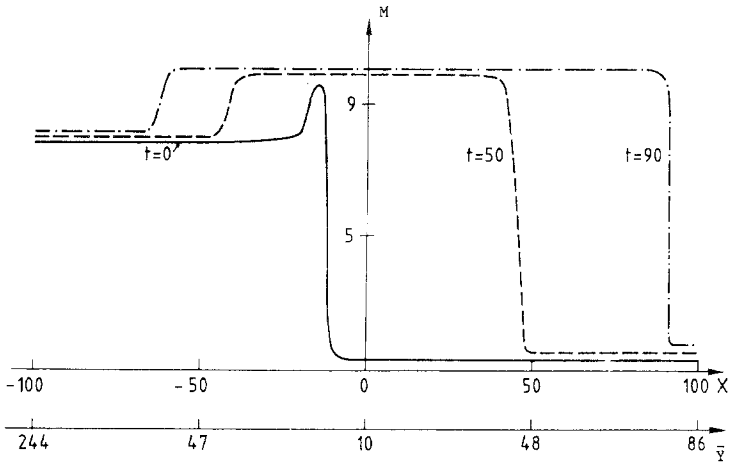
(a)

Fig. 3. (a) The M equidensity lines at $t = 0$. A bump is present in the shock domain. Lines parallel to $x = 0$ and to $\tau_3 \bar{y} + \bar{\gamma}_3 x = 0$ $(a = 1)$. (b, c) Movement of the shock. (b) For $\bar{y} = 0$, the curves rise continuously when t is growing. (c) For $x = 0$ and $|\bar{y}| \to \infty$ we recover the $t = 0$ shock limits. (d) M for $\tau_3(\bar{y} - 10) + \bar{\gamma}_3 x = 0$. Bump at $t = 0$, movement of the shock; asymptotic Σ^3 and Σ^{03} limits replaced by Σ^2 and m_0 $(a = 1)$.



(b) (c)

Fig. 3 *(continued)*



(d)

Fig. 3 (*continued*)

$d_j \to d_j \exp(\rho_j t_0)$ gives, for finite x, $\bar{y}$ values, a larger contribution for the positive first component $m_1 D_1^{-1}$ and a smaller one for the oher negative, components.

Figures 3b and 3c present both N_i and M relaxation curves for x, $\bar{y}$ along two lines. The first one, $\bar{y}=0$, at the bottom of the shock, is parallel to the shock front, while the other, $x=0$, is perpendicular to it. Along $\bar{y}=0$ we observe, when t is growing, a continuous rising of the curves up to equilibrium. The difference between the two $|x| \to \infty$ limits, which is equal to $m_3=0$ at $t=0$, falls progressively and disappears at equilibrium. Notice that these limits are t dependent, so that for t fixed we never recover the $t=0$ limits. Such a situation cannot arise in one spatial dimension. Along the profile $x=0$, perpendicular to the shock, we observe the moving of the shock, the small bump at the top of the shock front, and the spreading of the equilibrium state. Contrary to the previous case, for t fixed and $|\bar{y}|$ sufficiently large, we recover the asymptotic liits of the initial time.

Figure 3d presents a curve along a line $\tau_3(\bar{y}-10)+\bar{\gamma}_3 x=0$ for a large x (or $\bar{y}$) interval, parallel to one direction of the asymptotic plateaus. We observe the bump at $t=0$, the moving of the shock, and the appearance of the equilibrium state. The two asymptotic $|\bar{y}| \to \infty$ limits Σ^3 and m_0+m_3 at $t=0$ are progressively replaced by Σ^2 and m_0. This is explained by the displacement $-\rho_3 t<0$ of the equidensity lines $\tau_3\bar{y}+\bar{\gamma}_3 x=\text{const}$ toward the $x<0$ half-plane.

5. CONCLUSION

From the present work we know that positive $(2+1)$-dimensional shock waves exist for two discrete Boltzmann models. For the analytical positivity proof we were obliged to understand the mathematical structure of the asymptotic shock limits, which are physically relevant quantities. As a consequence of the laborious analytical calculation of Appendix C, we can now construct numerically positive shock waves for which the positivity has not been analytically proved. For the models of Section 3, giving up the restriction $S=-2(P+1)$ (leading to Theorems 2 and 3), we have constructed positive solutions.[9]

Taking advantage of the analytical results presented here, I am currently investigating two other classes of solutions: semiperiodic ones with the first two components complex conjugate, and solutions with six asymptotic shock limits.

APPENDIX A. SUFFICIENT ASYMPTOTIC POSITIVITY CONDITIONS

Theorem. Let

$$M=m_0+\sum_1^3 m_j D_j^{-1}, \qquad D_j=1+d_j e^{\tau_j y},$$

$$d_j>0, \qquad y \text{ real}, \qquad j=1,2, \qquad 0<D_3^{-1}<1$$

If one of the two conditions

$$\tau_1\tau_2>0, \qquad m_0>0, \qquad m_0+m_3>0, \qquad \sum_0^2 m_j>0, \qquad \sum_0^3 m_j>0 \tag{A.1}$$

$$\tau_1\tau_2<0, \qquad m_0+m_j+m_3>0, \qquad m_0+m_j>0, \qquad j=1,2 \tag{A.2}$$

is satisfied, then $M>0$ provided that the d_j satisfy sufficient conditions.

We remark that if $m_3>0$ (or <0), then $M>m_0$ (or m_0+m_3) $+\sum_1^2 m_j D_j^{-1}$, we must prove the following lemma.

Lemma. If $P=p_0+\sum_1^2 p_j D_j^{-1}$ and if one of the two conditions

$$\tau_1\tau_2>0, \qquad p_0>0, \qquad p_0+p_1+p_2>0$$

or (A.1′)

$$\tau_1\tau_2<0, \qquad p_0+p_j>0, \qquad j=1,2$$

is satisfied, then $P>0$ provided that the d_j satisfy sufficient conditions.

(i) Case $\tau_1\tau_2>0$: If $p_j>0$ (or $p_j<0$), $j=1,2$, then $P>p_0$ (or $p_0+p_1+p_2)u$ positive. It remains $p_2<0$, and we choose $p_1<0$, $p_2>0$, and assume $\tau_j>0$. We have

$$P=[p_0+p_1+p_2+(p_0+p_2)u_1+(p_0+p_1)u_2+p_0u_1u_2]/D_1D_2,\qquad u_j=d_je^{\tau_j y} \quad \text{(A.3)}$$

Only p_0+p_1 can be negative. If so, we find

$$\begin{aligned}&u_2(p_0u_1+p_1)>0 \qquad \text{if}\quad d_1>|p_1/p_0|\,e^{\tau_1 y_0} \quad \text{and} \quad y>-y_0,\quad y_0>0\\ &p_0+p_1+p_2+(p_1+p_0)u_2>0\\ &\qquad \text{if}\quad d_2<|(p_0+p_1+p_2)/(p_0+p_1)|\,e^{-\tau_2 y_0} \quad \text{and} \quad y\leqslant -y_0\end{aligned} \quad \text{(A.4)}$$

with y_0 fixed but arbitrary. Then $P>0$ for all y real values.

(ii) Case $\tau_1\tau_2<0$ and we assume $\tau_1>0$, $\tau_2<0$. If $p_0<0$, then $p_1>0$, $p_2>0$, $p_0+p_1+p_2>0$, only $p_0u_1u_2<0$ in (A.3), and we find

$$\begin{aligned}&u_1(p_0+p_2+p_0u_2)>0 \qquad \text{if}\quad d_2<|(p_2+p_0)/p_0| \quad \text{and} \quad y\geqslant 0\\ &u_2(p_0+p_1+p_0u_1)>0 \qquad \text{if}\quad d_1<|(p_0+p_1)/p_0| \quad \text{and} \quad y\leqslant 0\end{aligned} \quad \text{(A.5)}$$

If $p_0>0$, only $p_0+p_1+p_2$ can be negative in (A.3). If so, one p_j (or both) is negative,

$$\begin{aligned}&(p_0+p_2)(1+u_1)+p_1>0 \qquad \text{if}\quad d_1>|(p_0+p_2)/p_1|-1 \quad \text{and} \quad y\geqslant 0\\ &(p_0+p_1)(1+u_2)+p_2>0 \qquad \text{if}\quad d_2>|(p_0+p_1)/p_2|-1 \quad \text{and} \quad y\leqslant 0\end{aligned} \quad \text{(A.6)}$$

Finally, P is positive in both cases for all y values.

APPENDIX B. MODEL WITH THE FIRST TWO COMPONENTS DEPENDING ONLY ON y

B.1. Relations

The solutions

$$\begin{aligned}&N_i=n_{0i}+\sum_{j=1}^{3} n_{ji}D_j^{-1}, \qquad i=1,\dots,4\\ &D_j=1+d_j\exp(\tau_j y+\rho_j t), \qquad j=1,2\\ &D_3=1+d_3\exp(\tau_3 y+\rho_3 t+\gamma_3 x)\end{aligned}$$

with 23 parameters n_{0i}, n_{ji}, ρ_j, τ_j, γ_j substituted into the nonlinear discrete model Eq. (1.1) lead to 19 relations

$$n_{j1} = n_{j2}, \quad j=1,2, \qquad n_{01}n_{02} = an_{03}n_{04}, \qquad a(n_{14}n_{23} + n_{13}n_{24}) = 2n_{21}n_{11} \tag{B.1}$$

$$\begin{aligned} n_{j1}\rho_j = n_{j3}(\rho_j + \tau_j) = n_{j4}(\tau_j - \rho_j) &= an_{j3}n_{j4} - n_{j1}^2 \\ &= -a(n_{03}n_{j4} + n_{04}n_{j3}) + n_{j1}n_{21}^+ \end{aligned} \tag{B.2}$$

$$\begin{aligned} n_{31}(\rho_3 + \gamma_3) = n_{32}(\rho_3 - \gamma_3) = -n_{33}(\rho_3 + \tau_3) &= n_{34}(\tau_3 - \rho_3) \\ &= an_{33}n_{34} - n_{31}n_{32} \\ &= -a(n_{03}n_{34} + n_{04}n_{33}) + n_{01}n_{32} + n_{02}n_{31} \end{aligned} \tag{B.3}$$

$$a(n_{j4}n_{33} + n_{j3}n_{34}) = n_{j1}(n_{32} + n_{31}) \tag{B.4}$$

We have put $n_{21}^+ = n_{01} + n_{02}$. Relations (B.1), (B.2) are for the first two components $j = 1, 2$, while (B.3), (B.4) are those of the third one. Since a is not fixed, we have five free parameters.

B.2. Solutions

We define two new parameters $z_j = n_{j4}/n_{j3}$, $j = 1, 2$, and write $P = z_1 z_2$ and $S = z_1 + z_2$. We choose $(P, S, n_{01}, n_{02}, n_{03})$ as the arbitrary parameters.

B.2.1. Parameters for the First Two Components $j = 1, 2$. For simplicity we introduce intermediate parameters $\bar{n}_{j1} = n_{j1}/n_{j3}$ and from (B.1), (B.2) deduce

$$\bar{n}_{j1} = -2z_j/(1+z_j), \qquad a = 8P/[S(S+P+1)] \tag{B.5}$$

$$n_{j3}(az_j - \bar{n}_{j1}^2) = -a(n_{03}z_j + n_{04}) + \bar{n}_{j1}n_{21}^+ = \tau_j 2z_j/(1-z_j) \tag{B.6}$$

whence all n_{ji}, τ_j, ρ_j, and a are known:

$$\begin{gathered} n_{j3} = 2\{P[n_{03}(1+z_j) + 2n_{21}^+ a^{-1}] + n_{04}(z_i + P)\}/(z_j - z_i)(z_j - P), \qquad i \neq j \\ n_{j4} = z_j n_{j3}, \qquad n_{j1} = n_{j2} = -2z_j n_{j3}/(1+z_j) \end{gathered} \tag{B.7}$$

$$\begin{gathered} 2\tau_j z_j = (z_j - 1)[a(n_{03}z_j + n_{04}) + 2z_j n_{21}^+/(1+z_j)] \\ \rho_j = -\tau_j n_{j3}/(n_{j1} + n_{j3}) \end{gathered} \tag{B.8}$$

B.2.2. Parameters for the Third Component. We introduce other intermediate parameters $y_3 = n_{31}/n_{32}$ and $\bar{n}_{3i} = n_{3i}/n_{32}$, $i = 3, 4$, and obtain from (B.3), (B.4) their expressions as functions of the free parameters P, S.

$$y_3^2+2y_3[1-2(1+P)/S]+1=0 \rightarrow y_3^{\pm} = -B' \pm (B'^2-1)^{1/2}$$
$$B'=1-2(1+P)/S \tag{B.9}$$

$$\bar{n}_{33}= -y_3(1+P)/P(1+y_3)= -S(1+y_3)/4P, \qquad \bar{n}_{34}=P\bar{n}_{33}$$
$$(\bar{n}_{33}+\bar{n}_{34})\, y_3+\bar{n}_{33}\bar{n}_{34}(1+y_3)=0 \tag{B.10}$$

From (B.3), n_{32} can be written down with the intermediate parameters:

$$n_{32}(aP\bar{n}_{33}^2-y_3)= -a\bar{n}_{33}(n_{03}P+n_{04})+n_{01}+y_3 n_{02} \tag{B.11}$$

whence all the parameters n_{3i}, ρ_3, τ_3, γ_3 of the third component are known:

$$n_{32}(P+1-S)/(P+1+S)=n_{02}+n_{01}/y_3+a(P+1)(n_{03}+n_{04}/P)/(1+y_3),$$
$$n_{31}=y_3 n_{32}$$
$$n_{33}(P+1-S)/2(P+1)=n_{03}+n_{04}P^{-1}+(P+1+S)$$
$$\times (n_{01}+n_{02}\, y_3/2P(1+y_3) \tag{B.12}$$
$$=(n_{33}+n_{34})(n_{31}n_{32}-an_{33}n_{34})$$
$$n_{34}=Pn_{33}, \qquad \rho_3 2n_{33}n_{34}=(n_{33}+n_{34})(n_{31}n_{32}-an_{33}n_{34})$$
$$\tau_3(n_{33}+n_{34})=\rho_3(n_{34}-n_{33}), \qquad \gamma_3(n_{32}+n_{31})=\rho_3(n_{32}-n_{31})$$

with y_3 and a written in (B.5)–(B.9) as functions of P, S.

B.3. Determination of the Asymptotic Quantities Σ_i

We want to express the 12 quantities $\Sigma_1^2=\sum_{j=0}^2 n_{ji}$, $\Sigma_i^{03}=n_{0i}+n_{3i}$, $\Sigma_i^3=\sum_{j=0}^3 n_{ji}$ as functions of the free parameters P, S, n_{01}, n_{02}, and n_{03}. Invariance properties allow us to calculate explicitly only six of them.

B.3.1. Invariance Properties.

From the relations $n_{j1}=n_{j2}$, $j=1, 2$, $n_{31}/n_{32}=y_3$ we deduce

$$\Sigma_1 \leftrightarrow \Sigma_2 \quad \text{with the transform } (n_{01} \leftrightarrow n_{02},\ y_3 \leftrightarrow y_3^{-1}) \tag{B.13}$$

From the relations $n_{j3}/n_{j4}=z_j$, $j=1, 2$, we get

$$\Sigma_3 \leftrightarrow \Sigma_4 \quad \text{with the transform } (n_{03} \leftrightarrow n_{04},\ P \leftrightarrow P^{-1},\ S \leftrightarrow SP^{-1}) \tag{B.14}$$

However, the $n_{03}\Sigma_i$ are written as polynomials in n_{03} of the second degree with coefficients that are functions of P, S. From $n_{04}=n_{01}n_{02}/an_{03}$ we see that (B.14) is equivalent to

$$n_{03}\Sigma_3=\Omega_3(n_{03}+n_{01}A_{13})(n_{03}+n_{02}A_{23})\rightarrow n_{03}\Sigma_4$$
$$=\Omega^T A_{13}^T A_{23}^T a(n_{03}+n_{01}/aA_{23}^T)(n_{03}+n_{02}A_{13}^T a) \qquad \text{(B.14')}$$

where Ω_3^T means $\Omega(P\rightarrow P^{-1},\ S\rightarrow SP^{-1})$, $A_{12}^T=\cdots$.

For the calculated Σ_i we use the following method: Since all the n_{ji} are linear combinations of the n_{0i}, the same property holds for the Σ_i. One can write

$$n_{04}=n_{01}n_{02}S(S+P+1)/8Pn_{03} \qquad \text{(B.15)}$$

and then $n_{03}\Sigma_i$ is a second-degree polynomial in n_{03}. It turns out that the roots are n_{0j}, $j=1, 2$, multiplied by functions of P and S only. Further, all the roots are real.

B.3.2. Σ_i^2. From (B.7) we obtain the linear n_{0i} relations for Σ_1^2 and Σ_3^2:

$$(P+1-S)(n_{11}+n_{21})=4Pn_{03}+4n_{04}+2Sn_{21}^+$$
$$(P+1-S)\Sigma_1^2=4Pn_{03}+4n_{04}+2S_{02}+(P+1+S)n_{01}$$
$$(S-P-1)\sum_{j=1}^{2} n_{j3}=2n_{04}S/P+2(P+1)n_{03}+n_{21}^+S(S+P+1)$$
$$(S-P-1)\Sigma_3^2=(S+P+1)(n_{03}+n_{21}^+S/2P)+2n_{04}S/P$$

while (B.15) leads to the quadratic relations and the transforms (B.13)–(B.14') to Σ_2^2, Σ_4^2:

$$n_{03}\Sigma_1^2=\Omega_1^2(n_{03}-n_{01}A_1)(n_{03}-n_{02}A_2)$$
$$n_{03}\Sigma_2^2=\Omega_2^2(n_{03}-n_{01}A_2)(n_{03}-n_{02}A_1) \qquad \text{(B.16)}$$
$$\Omega_1^2=\Omega_2^2=4P/(P+1-S), \quad A_1=-(P+1+S)/4P, \quad A_2=-S/2P=1/aA_1$$
$$n_{03}\Sigma_3^2=\Omega_3^2(n_{03}-n_{01}A_2)(n_{03}-n_{02}A_2)$$
$$n_{03}\Sigma_4^2=\Omega_4^2(n_{03}-n_{01}A_1)(n_{03}-n_{02}A_1) \qquad \text{(B.17)}$$
$$\Omega_3^2=(1+P+S)/(S-1-P), \qquad \Omega_4^2=2S/P(S-1-P)$$

B.3.3. Σ_i^{03}. Adding to n_{32}, n_{33} [see (B.12)] either n_{02} or n_{03}, we get the linear relations for $\Sigma_{2\text{ or }3}^{03}$,

$$(P+1-S)\Sigma_2^{03} = 8(P+1)(n_{03}P+n_{04})/S(1+y_3) + (P+1+S)n_{01}/y_3 + 2(P+1)n_{02}$$

$$(S-P-1)\Sigma_3^{03}/(S+P-1) = n_{03} + 2n_{04}(P+1)/P(P+S+1) + (P+1)(n_{01}+n_{02}y_3)/P(1+y_3)$$

while (B.15), the identity $(1+y_3)^2 = y_3 4(P+1)/S$, and the transforms (B.13)–(B.14′) applied to Σ_i^{03}, $i=2, 3$, give Σ_i^{03}, $i=1$ and 4:

$$\begin{aligned}
n_{03}\Sigma_1^{03} &= \Omega_1^{03}(n_{03}-n_{01}A_3)(n_{03}-n_{02}A_4)\\
n_{03}\Sigma_2^{03} &= \Omega_2^{03}(n_{03}-\tilde{A}_4 n_{01})(n_{03}-\tilde{A}_3 n_{02})\\
\Omega_1^{03} &= 2P(1+y_3)/(1+P-S)\\
\Omega_2^{03} &= \Omega_1^{03}(y_3\to y_3^{-1}) = y_3^{-1}\Omega_1^{03}\\
A_3 &= -(P+1)/P(1+y_3)\\
\tilde{A}_3 &= A_3(y_3\to y_3^{-1}) = A_3 y_3\\
A_4 &= -(P+S+1)\,y_3/2P(1+y_3)\\
\tilde{A}_4 &= A_4(y_3\to y_3^{-1}) = A_4/y_3
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
n_{03}\Sigma_3^{03} &= \Omega_3^{03}(n_{03}-n_{01}A_3)(n_{03}-n_{02}\tilde{A}_3)\\
n_{03}\Sigma_4^{03} &= \Omega_4^{03}(n_{03}-n_{01}\tilde{A}_4)(n_{03}-n_{02}A_4)
\end{aligned} \tag{B.19}$$

$$\Omega_3^{03} = (P+S+1)/(S-P-1), \qquad \Omega_4^{03} = 2P(P+1)/(S-P-1)$$

B.3.4. Σ_i^3. We need other y_3 identities:

$$\begin{aligned}
(3+y_3)^2 &= 4[2+y_3(1+P+S)/S]\\
S(S+3P+3) &= [S+2(P+1)/(1+y_3)][S+2(P+1)\,y_3/(1+y_3)]
\end{aligned} \tag{B.9′}$$

To the linear n_{0i} relations Σ_i^2, $i=1, 3$, of Section B.3.2 we add, respectively, n_{31} and n_{32}:

$$(P+1-S)\Sigma_1^3 = 2P(3+y_3)n_{03} + 2(3+y_3)n_{04} + [2S+(P+1+S)\,y_3]n_{02} + 2(P+S+1)n_{01}$$

$$(S-P-1)\Sigma_3^3 = n_{03}(S+3P+3) + \frac{2n_{04}(S+P+1)}{P} + \frac{1+P+S}{2P} \times\left[n_{01}\left(S+\frac{2P+2}{1+y_3}\right) + n_{02}\left(S+\frac{2(P+1)\,y_3}{1+y_3}\right)\right]$$

With (B.15)–(B.9′) we find the quadratic relations and with the transforms (B.13)–(B.14′) deduce Σ_2^3, Σ_4^3:

$$\begin{aligned}
n_{03}\Sigma_1^3 &= \Omega_1^3(n_{03}-n_{01}A_5)(n_{03}-n_{02}A_6)\\
n_{03}\Sigma_2^3 &= \Omega_2^3(n_{03}-n_{01}\tilde{A}_6)(n_{03}-n_{02}\tilde{A}_5)\\
-A_5 &= (P+S+1)/P(3+y_3)\\
-A_6 &= [2S+y_3(P+1+S)]/2P(3+y_3)\\
\tilde{A}_5 &= A_5(y_3\to y_3^{-1})\\
\tilde{A}_6 &= A_6(y_3\to y_3^{-1})
\end{aligned} \tag{B.20}$$

$$\Omega_1^3=2P(3+y_3)/(P+1-S), \qquad \Omega_2^3=\Omega_1^3(y_3\to y_3^{-1})$$

$$\begin{aligned}
n_{03}\Sigma_3^3 &= \Omega_3^3(n_{03}-n_{01}A_5)(n_{03}-n_{02}\tilde{A}_5)\\
n_{03}\Sigma_4^3 &= \Omega_4^3(n_{03}-n_{01}\tilde{A}_6)(n_{03}-n_{02}A_6)
\end{aligned} \tag{B.21}$$

$$\Omega_3^3=(3+3P+S)/(S-P-1), \qquad \Omega_4^3=2P(S+P+1)/(S-P-1)$$

B.4. Sufficient Positivity Conditions for the Σ_i

We define a scaling parameter $S=-s(P+1)$, and

$$\bar{n}_{03}=n_{03}P/(P+1), \qquad A_i=(P+1)B_i/P, \qquad \tilde{A}_i=(P+1)\tilde{B}_i/P, \qquad \bar{\Sigma}_i=\bar{n}_{03}\Sigma_i(s+1) \tag{B.22}$$

The above relations for $n_{03}\Sigma_i$ become

$$\bar{\Sigma}_i=\Omega_i(P+1)\,P^{-1}(\bar{n}_{03}-B_{1i}n_{01})(\bar{n}_{03}-B_{2i}n_{02}) \tag{B.23}$$

where the roots B_{1i} and B_{2i} are obviously the B_i and $\tilde{B}_i$ deduced from (B.16)–(B.21) as written down in Table I.

Lemma 1. Let

$$B_1=(s-1)/4, \qquad B_2=s/2, \qquad B_3=-(1+y_3)^{-1}$$

$$B_4=(s-1)\,y_3/2(1+y_3), \qquad B_5=(s-1)/(3+y_3)=2(s-1)\tilde{B}_6/(s-3)$$

$$B_6=[2s+(s-1)\,y_3]/2(3+y_3)=[-2y_3+s(1+y_3)]/4(1+y_3)$$

$$\tilde{B}_3=B_3\,y_3, \qquad \tilde{B}_4=B_4/y_3$$

$$\tilde{B}_5=(s-1)\,y_3/(3y_3+1), \qquad \tilde{B}_6=[s(1+y_3)-2]/4(1+y_3)$$

If

$$s>3:\quad y_3=y_3^-=-\{1+2[1+(s+1)^{1/2}]/s\}) \tag{B.24}$$

then the following properties hold:

$$\text{(i)}\quad y_3+1<0<y_3+3,\quad B_i>0,\quad i=1,\dots,6$$

$$\tilde{B}_5>0,\quad \tilde{B}_6>0,\quad \tilde{B}_3<0,\quad \tilde{B}_4<0,\quad s+y_3(s-2)>0$$

$$\text{(ii)}\quad B_6<B_1<\tilde{B}_5,\quad B_1<\tilde{B}_6<B_5$$

$$\text{(iii)}\quad B_1<B_2,\quad B_3<B_1<B_4$$

Proofs. (i) $y_3+1<0$ is obvious; $(y_3+3)s/2=s-1-(s+1)^{1/2}>0$ is equivalent to $s(s-3)>0$ and

$$-2y_3+s(1+y_3)=(2/s)[2+(2-s)(s+1)^{1/2}]<(2/s)[2-(s+1)^{1/2}]\leqslant 0$$

Consequently all B_i, $\tilde{B}_5$, $\tilde{B}_6$ are positive, while $\tilde{B}_3$, $\tilde{B}_4$ are negative.

(ii)
$$-1+B_6/B_1=(y_3-1)/(1+y_3)(1-s)<0$$
$$-1+\tilde{B}_5/B_1=(y_3-1)/(3y_3+1)>0$$
$$-1+B_5/\tilde{B}_6=(s+1)/(s-3)>0$$

Also note the relations

$$2B_1=B_6+\tilde{B}_6=B_4+\tilde{B}_4$$
$$B_3+\tilde{B}_3=-1\to -1+\tilde{B}_6/B_1=1-B_6/B_1>0$$

(iii)
$$B_1/B_2=1/2-1/2s<1/2$$
$$-1+B_1/B_3=(s+1)^{1/2}[s-1-(s+1)^{1/2}]/2s>0$$
$$B_1/B_4=1/2+1/2y_3<1/2$$

Theorem 1. All the Σ_i are positive if the following sufficient conditions are satisfied:

$$\begin{gathered} s>3,\qquad P>0,\qquad y_3=y_3^-,\qquad 0<b_{01}<n_{02}B_6/B_1 \\ n_{01}B_3<\bar{n}_{03}=n_{03}P/(P+1)<n_{01}B_1 \end{gathered} \tag{B.25}$$

Proofs. For Σ_i^2. All the coefficients of $\bar{n}_{03}^2$ as well as the roots $n_{0k}B_j$, $j,k=1,2$, $j\neq k$ are positive. It is sufficient for $\Sigma_i^2>0$ that $\bar{n}_{03}$ be less than the inf of the roots. From the lemma $n_{01}<n_{02}B_6/B_1<n_{02}$ and $B_1<B_2$. The smallest root is $n_{01}B_1$ and $\Sigma_i^2>0$ if $\bar{n}_{03}<n_{01}B_1$.

For Σ_i^{03}. For Σ_2^{03} the coefficient of $\bar{n}_{03}$ is positive and the two roots proportional to B_3 and B_4 are negative. $\Sigma_2^{03}>0$ for $\bar{n}_{03}>0$. For Σ_1^{03} the coefficient of $\bar{n}_{03}^2$ is negative and the two roots are positive. For the positivity, applying the lemma, it is sufficient that $n_{01}B_3<\bar{n}_{03}<n_{02}B_4$. For

Σ_3^{03} the coefficient of $\bar{n}_{03}^2$ is positive, one root is positive, and the other is negative. For $\Sigma_3^{03} > 0$ then $\bar{n}_{03} > n_{01} B_3$. For Σ_4^{03} the coefficient of $\bar{n}_{03}^2$ is negative, with one root positive and the other negative; for positivity it is sufficient that $\bar{n}_{03} < n_{02} B_4$. Due to $B_3 < B_1 < B_4$ we see that both Σ_i^2 and Σ_i^{03} are positive with (B.25).

For Σ_i^3. All four coefficients of $\bar{n}_{03}^2$ are positive and the four roots $n_{01} B_5$, $n_{01} \tilde{B}_6$, $n_{02} B_6$, and $n_{02} \tilde{B}_5$ are positive. For positivity it is sufficient that $\bar{n}_{03}$ be less than the smallest root. From the lemma and the hypothesis (B.25), $n_{01} B_1$ is smaller than all roots and $\Sigma_i^3 > 0$ for $\bar{n}_{03} < n_{01} B_1$. In conclusion, (B.25) is sufficient for the positivity of all Σ_i. Finally, we notice that $z_j = z_\pm = (S \mp \sqrt{\Delta})/2$ are real and negative for $s > 3$ and $P > 0$ because $\Delta = S^2 - 4P > 0$ and $S = -s(P+1) < 0$,

$$2z_\pm = s(P+1)(-1 \mp \sqrt{\delta}), \qquad \delta = 1 - 4P[s(P+1)]^{-2} \tag{B.26}$$

From $1 + z_+ = x + [x^2 + (P+1)(s-1)]^{1/2}$ with $x = 1 - s(P+1)/2 < 0$, it follows that $1 + z_+ > 0$, while $S < -3$, $z_- < -2$.

B.5. Condition $\tau_1 \tau_2 > 0$

The sign of $\tau_1 \tau_2$ is given [see (B.8)] by the product of two quadratic polynomials in n_{03}

$$\begin{gathered} \tau_1 \tau_2 n_{03}^2 / a^2 (P+1) = \mathscr{T}_1 \mathscr{T}_2, \qquad \mathscr{T}_i = n_{03}^2 + 2 n_{03} \alpha/(1+z_i) + \beta/z_i \\ \alpha = n_{21}^+/a, \qquad \beta a = n_{01} n_{02}, \qquad a\alpha^2 > 4\beta \end{gathered} \tag{B.27}$$

The two roots $n_{03,z}^\mp$ of the polynomial $\mathscr{T}_i$ are real and have opposite signs ($\beta z_i < 0$). Then $\tau_1 \tau_2 > 0$ if, for instance, $0 < n_{03} < \inf(n_{03,z_\pm})$, where the positive roots are

$$n_{03,z_\pm} = -\alpha/(1+z_\pm) + \sqrt{\Delta_\pm} > 0, \qquad \Delta_\pm = [\alpha/(1+z_\pm)]^2 - \beta/z \tag{B.28}$$

First we show that n_{03,z_+} is the smallest root and second that $\tau_1 \tau_2 > 0$ for (B.25).

Lemma 2. If $P > 0$, $s > 3$ we find the inequalities: (i) $\Delta_+ > \Delta_-$, (ii) $\Delta_+ + \Delta_- < \alpha^2 \Delta/(1+P+S)^2$, (iii) $n_{03,z_+} < n_{03,z_-}$.

Proofs. (i) We notice that $\sqrt{\Delta} = z_+ - z_-$ and find

$$\Delta_+ - \Delta_- = \sqrt{\Delta}[\beta/P - \alpha^2(S+2)/(1+P+S)^2] > 0$$

because $S + 2 < 0$, $\alpha > 0$, $\beta > 0$.

(ii) We find

$$\Delta_+ + \Delta_- - \alpha^2 \Delta/(1+P+S)^2 = 2(\alpha^2 - 4\beta/a)/(1+P+S) < 0$$

due to $1+P+S=(1+P)(1-s)<0$.

(iii) First we have $(\sqrt{\Delta_+}-\sqrt{\Delta_-})^2 < \Delta_+ + \Delta_- < \alpha^2\Delta/(1+P+S)^2$ and taking the positive determination of the square roots we find $\sqrt{\Delta_+}-\sqrt{\Delta_-}-\alpha\sqrt{\Delta}/(1+P+S)<0$ or $n_{03,z_+}<n_{03,z_-}$.

For the solutions satisfying Theorem 1 [Eq. (B.25)],

$$\text{if} \quad n_{03,z_+} > (P+1)n_{01}B_1/P \text{ and } 0 < n_{03} < n_{03,z_+} \quad \text{then} \quad \tau_1\tau_2 > 0 \tag{B.29}$$

Lemma 3. We define

$$Q = Q_1 + Q_2$$

$$Q_1 = n_{01}B_1(P+1)[aB_1(P+1)/P + 2/(1+z_+)]$$

$$Q_2 = n_{02}[z_- + 2B_1(P+1)/(1+z_+)]$$

Then (B.29) is satisfied if $Q<0$.

We remark that

$$Qn_{01}/aP = -\Delta_+ + [\alpha/(1+z_+) + B_1(P+1)n_{01}/P]^2$$

with Δ_+, α, β given by $(B.27)$–(B.28) and if $Q<0$ then $\sqrt{\Delta_+} > \alpha/(1+z_+) + B_1 n_{01}(P+1)/P$ or equivalently $n_{03,z_+} > (P+1)n_{01}B_1/P$,

Lemma 4. $Q_1>0$, $Q_2<0$. We recall that $1+z_+>0$ [(B.26)] and obtain $2Q_2(1+z_+)(P+1)n_{02} = x - \sqrt{\delta} < 0$ because $x=(P-1)s^{-1}/(P+1)$ and $\delta = x^2 + 1 - s^{-2} > x^2$. On the other hand,

$$Q_1/n_{01}B_1(P+1) = 2(P+1)/s + (1+\sqrt{\delta})/(1+z_+) > 0$$

Consequently, if in (B.25) $n_{01}=0$, then $Q<0$ and this property holds for any n_{01} if it holds for n_{01} $\sup = n_{02}B_6/B_1$.

Lemma 5. $Q<0$ for $n_{01}=n_{01}$ sup. From Lemma 3 we have

$$2(1+z_+)\,Q/n_{02}(P+1) < \bar{Q} = (P-1)/(P+1) - s\sqrt{\delta} + 2B_6[1+\sqrt{\delta}+2/s(P+1)]$$

$$\bar{Q} = \bar{Q}_1 + \bar{Q}_2, \qquad y_3 = y_3^- \tag{B.30}$$

$$\bar{Q} = 1 + s - (s+y_3)[s(1+\sqrt{\delta}) + 2/(P+1)]$$

$$\bar{Q}_1 = [2-(s+1)^{1/2}][1/(P+1) + s(1+\sqrt{\delta})/2]/[s-(s+1)^{1/2}-1] < 0$$

$$\bar{Q}_2 = (s+2)(x-\sqrt{\delta})$$

with

$$x = s/(s+2) - 2/s(P+1)$$
$$\delta = x^2 + 4[(s+1)/(s+2)^2 + (s^2 - s - 2)/s^2(P+1)(s+1)]$$

Due to $\delta - x^2 > 0$, we find $\bar{Q}_2 < 0$, leading to $\bar{Q} < 0$ and $Q < 0$.

Theorem 1bis. Because the conditions (B.25) on the five arbitrary parameters lead to N_i solutions with $\tau_1\tau_2 > 0$, then for these solutions their asymptotic positivity conditions $\Sigma_i > 0$ are satisfied.

APPENDIX C. MODELS WITH THE TWO FIRST COMPONENTS DEPENDING ONLY ON $y + \mu x$ AT $t = 0$

C.1. Relations

The solutions

$$N_i = n_{0i} + \sum_{j=1}^{3} n_{ji} D_j^{-1}$$
$$D_j = 1 + d_j \exp(\tau_j y + \gamma_j x + \rho_j t), \qquad i = 1,..., 4, \quad \gamma_j = \tau_j \mu, \quad j = 1, 2$$

with 26 parameters n_{0i}, n_{ji}, τ_j, γ_j, ρ_j, and μ substituted into the nonlinear discrete model Eq. (1.1) lead to 21 relations: $a = 1$ and

$$\begin{aligned}
&\gamma_j = \mu\tau_j, \quad j = 1, 2, \qquad n_{04} = n_{01}n_{02}/n_{03} \\
&n_{p3}n_{m4} + n_{p4}n_{m3} = n_{p1}n_{m2} + n_{p2}n_{m1}, \qquad p \neq m \\
&n_{j1}(\rho_j + \gamma_j) = n_{j2}(\rho_j - \gamma_j) = -n_{j3}(\rho_j + \tau_j) = n_{j4}(\tau_j - \rho_j) \\
&\quad = n_{j3}n_{j4} - n_{j1}n_{j2} = n_{01}n_{j2} + n_{02}n_{j1} - n_{03}n_{j4} - n_{04}n_{j3}, \qquad j = 1, 2, 3
\end{aligned} \tag{C.1}$$

C.2. Solutions

We again define $z_j = n_{j4}/n_{j3}$, $j = 1, 2$, $P = z_1 z_2$, $S = z_1 + z_2$, and choose $(P, S, n_{0j}, j = 1, 2, 3)$ as the five arbitrary parameters from which we deduce the others. We note that n_{04} is obtained from (C.1).

C.2.1. Parameters for the Two First Components $j = 1, 2$. We again introduce intermediate parameters $\bar{n}_{ji} = n_{ji}/n_{j3}$ and from (C.1) deduce

$$\begin{aligned}
&\bar{n}_{j1} = 2z_j/C_j, \qquad C_j = \mu - 1 - (\mu + 1)z_j \\
&\bar{n}_{j2} = 2z_j/E_j, \qquad E_j = C_j(-\mu) = -\mu - 1 + (\mu - 1)z_j
\end{aligned} \tag{C.2}$$

with $j=1,2$. At this stage μ is unknown; however, the compatibility relation $p, m=1, 2$ in (C.1) becomes $z_1+z_2=\bar{n}_{12}\bar{n}_{21}+\bar{n}_{22}\bar{n}_{11}$, and leads to $\mu(P, S)$ and so to $\bar{n}_{ji}(P, S)$:

$$\mu^2=(1+P+S)/(1+P-S)^2-8P(1+P+S)/S(1+P-S)^2 \qquad \text{(C.3)}$$

The rhs of (C.3) must be positive and μ has two possible determinations. From the definitions of z_j and $\bar{n}_{ji}$ we see that the n_{ji} are known (as functions of the arbitrary parameters) once n_{j3} is obtained. From (C.1) and the n_{ji} we get ρ_j, τ_j, and γ_j:

$$\begin{aligned} n_{j3}&=M_j/(z_j-\bar{n}_{j1}\bar{n}_{j2}) \\ M_j&=-n_{03}z_j-n_{04}+n_{01}\bar{n}_{j2}+n_{02}\bar{n}_{j1} \\ n_{j4}&=z_j n_{j3}, \qquad n_{ji}=\bar{n}_{ji}n_{j3}, \qquad i=1,2 \end{aligned} \qquad \text{(C.4)}$$

$$\tau_j=(1-z_j)M_j/2z_j, \qquad \rho_j=\tau_j(1+z_j)/(z_j-1), \qquad \gamma_j=\mu\tau_j, \qquad j=1,2$$

C.2.2. Parameters for the Third Component $j=3$. We introduce intermediate parameters $y_3=n_{31}/n_{32}$ and $\bar{n}_{3i}/n_{32}$, $i=3, 4$, and the $p, m=1, 3$ and $2, 3$ (C.1) relations become $\bar{n}_{33}z_j+\bar{n}_{34}=\bar{n}_{j1}+y_3\bar{n}_{j2}$, leading to

$$\begin{aligned} \bar{n}_{33}&=2[(\mu-1)/C_1C_2-(\mu+1)\,y_3/E_1E_2] \\ \bar{n}_{34}&=-2P[(\mu+1)/C_1C_2-(\mu-1)\,y_3/E_1E_2] \\ &(\bar{n}_{33}+\bar{n}_{34})\,y_3+\bar{n}_{34}\bar{n}_{33}(1+y_3)=0 \end{aligned} \qquad \text{(C.5)}$$

and a cubic y_3 equation

$$\begin{aligned} &(y_3+1)\{2(\mu^2-1)[C_1C_2y_3^2/E_1E_2+E_1E_2/C_1C_2]-4y_3(\mu^2+1)\} \\ &\quad +y_3^2C_1C_2[1-\mu+(\mu+1)/P]+y_3E_1E_2[1+\mu+(1-\mu)/P]=0 \end{aligned} \qquad \text{(C.6)}$$

In (C.6) all coefficients C_j, E_j, μ are known P, S functions; consequently, (C.6) gives y_3 and (C.5) $\bar{n}_{3i}$ also as P, S functions. Now the construction of the n_{3i} parameters is possible once n_{32} is obtained:

$$\begin{aligned} n_{32}&=(n_{03}\bar{n}_{34}+n_{04}\bar{n}_{33}-n_{01}-n_{02}\,y_3)/(y_3-\bar{n}_{33}\bar{n}_{34}) \\ n_{31}&=y_3n_{32}, \qquad n_{3i}=\bar{n}_{3i}n_{32}, \qquad i=1,2 \end{aligned} \qquad \text{(C.7)}$$

Finally, the same relations as in (B.12) hold for τ_3, γ_3, and ρ_3

$$\begin{aligned} 2\rho_3n_{33}n_{34}&=(n_{33}+n_{34})(n_{31}n_{32}-n_{33}n_{34}) \\ \tau_3(n_{33}+n_{34})&=\rho_3(n_{34}-n_{33}) \\ \gamma_3(n_{32}+n_{31})&=\rho_3(n_{32}-n_{31}) \end{aligned} \qquad \text{(C.8)}$$

C.3. Determination of the Asymptotic Quantities Σ_i

As in Appendix B, two important properties exist: (i) the roots of $\Sigma_i = 0$ are of the type $n_{03} = n_{0j}$ multiplied by a function of P, S alone; (ii) there exist relations between the roots corresponding to different i values.

At the linear n_{0i} level of the relations if an identity holds, then (i) holds at the quadratic n_{03} level of the relations:

$$\Sigma_i = \Omega_i(n_{03} + \sum_{j \neq 3} n_{0j}\alpha_{ij}(P, S)$$

If $\alpha_{1i}\alpha_{2i} = \alpha_{4i}$, then

$$\Sigma_i = \Omega_i(n_{03} + n_{01}\alpha_{1i})(n_{03} + n_{02}\alpha_{2i})/n_{03} \tag{C.9}$$

These identities are trivial for Σ_i^{03} and difficult to prove for Σ_i^2 and Σ_i^3. We begin with the trivial case.

C.3.1. Σ_i^{03}.

From (C.7) we remark that the Σ_i are linear combination of the n_{0i}; we quote $\Sigma_i^{03}/\Omega_i^{03} - n_{03}$:

$$\begin{aligned}
i=1:&\quad n_{04}\bar{n}_{33}/\bar{n}_{34} - n_{01}\bar{n}_{34}/y_3 - n_{02}\, y_3/\bar{n}_{34}\\
i=2:&\quad n_{04}\bar{n}_{33}/\bar{n}_{34} - n_{01}/\bar{n}_{34} - n_{02}\bar{n}_{33}\\
i=3:&\quad n_{04}\bar{n}_{33}^2/y_3 - n_{01}\bar{n}_{33}/y_3 - n_{02}\bar{n}_{33}\\
i=4:&\quad n_{04}\, y_3/\bar{n}_{34}^2 - n_{01}/\bar{n}_{34} - n_{02}\, y_3/\bar{n}_{34}
\end{aligned} \tag{C.10}$$

Since the coefficients of n_{04} are the product of those for n_{01} and n_{02}, we apply (C.9) and find the quadratic n_{03} polynomials for $\Sigma_i n_{03}$:

$$\begin{aligned}
n_{03}\Sigma_1^{03} &= \Omega_1^{03}(n_{03} - A_3 n_{01})(n_{03} - A_4 n_{02})\\
n_{03}\Sigma_2^{03} &= \Omega_2^{03}(n_{03} - \tilde{A}_4 n_{01})(n_{03} - \tilde{A}_3 n_{02})\\
n_{03}\Sigma_3^{03} &= \Omega_2^{03}(n_{03} - A_3 n_{01})(n_{03} - \tilde{A}_3 n_{02})\\
n_{03}\Sigma_4^{03} &= \Omega_4^{03}(n_{03} - \tilde{A}_4 n_{01})(n_{03} - A_4 n_{02})
\end{aligned} \tag{C.11}$$

$$A_3 = \bar{n}_{33}/y_3, \qquad A_4 = y_3/\bar{n}_{34}, \qquad \tilde{A}_3 = \bar{n}_{33}, \qquad \tilde{A}_4 = 1/\bar{n}_{34},$$

$$\Omega_2^{03} = \bar{n}_{34}/(y_3 - \bar{n}_{33}\bar{n}_{34}), \quad \Omega_1^{03} = y_3\Omega_2^{03}, \quad \Omega_4^{03} = \bar{n}_{34}\Omega_2^{03}, \quad \Omega_3^{03} = \Omega_2^{03}\, y_3/\bar{n}_{34}$$

In fact, with the help of invariance properties, it was sufficient to calculate Σ_i^{03} for $i = 1$ and 3, then deduce Σ_i for $i = 2$ and 4:

(i) The relations $n_{31}/n_{32} = y_3$ and $n_{3i}/n_{31} = \bar{n}_{3i}\, y_3^{-1}$ with the exchange

$1 \leftrightarrow 2$ become $n_{32}/n_{31} = y_3^{-1}$ and $n_{3i}/n_{32} = \bar{n}_{3i}$. Let us consider P, S and y_3, $\bar{n}_{3i}$ as independent variables and get

$$\Sigma_1^{03} \to \Sigma_2^{03} \text{ with transform } (n_{01} \leftrightarrow n_{02},\ y_3 \to y_3^{-1},\ \bar{n}_{3i}/y_3 \to \bar{n}_{3i}) \qquad \text{(C.12)}$$

We easily verify with this transform that the roots $n_{01}A_3$ and $n_{02}A_4$ of Σ_1^{03} become the roots $n_{01}\tilde{A}_4$ and $n_{02}\tilde{A}_3$ of Σ_2^{03}, while Ω_1^{03} becomes Ω_2^{03}.

(ii) For the exchange $3 \leftrightarrow 4$ we see that $\bar{n}_{3i} \leftrightarrow \bar{n}_{4i}$ and deduce

$$\Sigma_3^{03} \to \Sigma_4^{03} \text{ with transform } (n_{03} \leftrightarrow n_{04},\ \bar{n}_{3i} \leftrightarrow \bar{n}_{4i}) \qquad \text{(C.13)}$$

For instance, for Σ_3^{03} the root $n_{03} = n_{01}A_3$ becomes $n_{04} = n_{01}A_3(\bar{n}_{33} \to \bar{n}_{34}) = n_{01}/A_4$ or $n_{03} = n_{02}A_4$ root of Σ_4^{03}. Similarly, the root $n_{03} = n_{02}\tilde{A}_3$ for Σ_3 becomes $n_{04} = n_{02}\tilde{A}_3(\bar{n}_{33} \to \bar{n}_{34}) = n_{02}/\tilde{A}_4$ or $n_{03} = n_{01}\tilde{A}_4$ root of Σ_4.

C.3.2. Σ_i^2. We write down the useful formulas

$$\begin{aligned} C_1C_2E_1E_2 &= 16P(S+P+1) \to (S^2-4P)/S^2(S-P-1) \\ (1-\mu^2)(1+P-S)^2 &= -4S(P+1) + 8P(1+P+S)/S \\ E_1E_2 &= 2(1+P+S)[S(P+1)-4P]/ \\ &\quad S(1+P-S) - 2\mu(P-1) \\ C_1C_2 &= E_1E_2(\mu \to -\mu) \end{aligned} \qquad \text{(C.14)}$$

For simplicity we put $\alpha_{ki} = \lambda_{ki}/\lambda_{3i}$ and rewrite (C.9):

$$\Sigma_i = \Omega_i\Big(n_{03} + \sum_{k \neq 3} n_{0k}\lambda_{ki}/\lambda_{3i}\Big)$$

If $\lambda_{1i}\lambda_{2i} = \lambda_{3i}\lambda_{4i}$, then

$$\Sigma_i = \Omega_i[n_{03} + (\lambda_{1i}/\lambda_{3i})n_{01}][n_{03} + (\lambda_{2i}/\lambda_{3i})n_{02}]/n_0 \qquad \text{(C.15)}$$

a. Σ_i^2, $i = 1, 2$. First for Σ_1^2 we use the expression (C.2)–(C.4) for n_{j1} and obtain:

$$\begin{aligned} \lambda_{31} &= [\mu(P+1-S) + S + P + 1](S-2P) + 4P(P-1) \\ \lambda_{41} &= [\mu(S-P-1) + S + P + 1](S-2) + 4(1-P) \\ \lambda_{11} &= 4P(1+P-S)/S \\ \lambda_{21} &= 2(4P-S^2)E_1E_2/C_1C_2 \\ \Omega_1^2 &= 2\lambda_{31}/(1-\mu^2)(1+P-S)^2 \end{aligned} \qquad \text{(C.16)}$$

Now we prove the identity (C.15). We find

$$\lambda_{11}\lambda_{21}=(E_1E_2)^2\,(S-P-1)^2\,S/2(S+P+1)$$

In (C.14), E_1E_2 contains P, S terms and terms proportional to μ. For the square, μ^2 becomes S, P dependent with (C.3), but terms proportional to μ remain:

$$\begin{aligned}\lambda_{11}\lambda_{12}/4=&(1+P+S)[S(P^2+1)-4P(P+1)+8P^2/S]\\&-4P(P-1)^2+\mu(1-P)(1+P-S)[S(P+1)-4P]\end{aligned}\qquad\text{(C.17)}$$

For the calculation of $\lambda_{31}\lambda_{41}/4$, from (C.16) we still find terms proportional to μ and others only S, P dependent. For both terms we identify with (C.17).

Second, for the exchange $\Sigma_1^2\leftrightarrow\Sigma_2^2$ we remark from (C.2) that $n_{j1}\leftrightarrow n_{j2}$ or $C_j\leftrightarrow E_j$ or $\mu\leftrightarrow-\mu$. We find finally in both cases

$$\begin{gathered}n_{03}\Sigma_1^2=\Omega_1^2(n_{03}-n_{01}A_1)(n_{03}-n_{02}A_2)\\ n_{03}\Sigma_2^2=\Omega_2^2(n_{03}-n_{01}\tilde{A}_2)(n_{03}-n_{02}\tilde{A}_1)\\ A_1=-\lambda_{11}/\lambda_{31},\qquad A_2=-\lambda_{21}/\lambda_{31},\qquad \tilde{A}_k=A_k(\mu\to-\mu)\\ \Omega_1^2=2\lambda_{31}/(1-\mu^2)(1+P-S)^2,\qquad \Omega_2^2=\Omega_1^2(\mu\to-\mu)\end{gathered}\qquad\text{(C.18)}$$

b. Σ_i^2, $i=3, 4$. First, for Σ_3^2, with the help of (C.4) for n_{j3} we find

$$\begin{gathered}\lambda_{33}=-\lambda_{11},\qquad \lambda_{23}=-\lambda_{41},\qquad \lambda_{13}=\lambda_{23}(\mu\to-\mu)=-\lambda_{41}(\mu\to-\mu)\\ \lambda_{43}=2(S^2-4P)/P,\qquad \Omega_3^2=2\lambda_{33}/(1-\mu^2)(1+P-S)^2\end{gathered}\qquad\text{(C.19)}$$

With (C.19) we prove the identity $\lambda_{13}\lambda_{23}=\lambda_{33}\lambda_{34}$. We find

$$\lambda_{13}\lambda_{23}=-\mu^2(S-P-1)^2\,(S-2)^2+[(S-2)(S+P+1)+4(1-P)]^2$$

and substituting (C.3) for μ^2, we can identify with $\lambda_{33}\lambda_{34}=8(S^2-4P)(S-P-1)/S$. Consequently, the quadratic representation (C.15) holds and the roots are $n_{03}=-n_{0j}\lambda_{j3}/\lambda_{33}$.

We prove that $n_{02}A_2$ is a common root to Σ_i^2, $i=1$ and 3. Using the identity (C.15) for $i=1$ and (C.19) we get

$$\lambda_{23}/\lambda_{33}=-\lambda_{23}/\lambda_{11}=\lambda_{41}/\lambda_{11}=\lambda_{21}/\lambda_{31}=-A_2$$

Finally, from the relation $\lambda_{13}/\lambda_{33}=\lambda_{23}/\lambda_{33}$ (with $\mu\to-\mu$) we see that the other root is $n_{01}\tilde{A}_2=n_{01}A_2(\mu\to-\mu)$. Second, for Σ_4^2 with (C.4) for n_{j4} we find

$$\begin{gathered}\lambda_{24}=\lambda_{14}(\mu\to-\mu)=-\lambda_{31},\qquad \lambda_{34}=2(S^2-4P)P\\ \lambda_{44}=-\lambda_{11},\qquad \Omega_4^2=2\lambda_{34}/(1-\mu^2)(S-P-1)^2\end{gathered}\qquad\text{(C.20)}$$

In the transform $3 \to 4$ with $n_{03} \leftrightarrow n_{04}$ and $n_{j3} \to n_{j4}$, n_{01} and n_{02} are not changed. This means that for the product of the two factors $n_{03} - n_{0j} A_k$ in (C.15), only n_{0j}, $j = 1, 2$, do not change and we still have a product of two similar factors. Consequently, the identity $\lambda_{14}\lambda_{24} = \lambda_{34}\lambda_{44}$ necessarily holds.

In order to prove that Σ_1^2 and Σ_4^2 have the common root $n_{01} A_1$, it is sufficient to notice that

$$\lambda_{14}/\lambda_{34} = \lambda_{44}/\lambda_{24} = \lambda_{11}/\lambda_{31} = -A_1$$

Finally, from $\lambda_{24}/\lambda_{34} = \lambda_{14}/\lambda_{34}$ (with $\mu \to -\mu) = -A_1$, we see that the other root for Σ_4^2 is $n_{02}\tilde{A}_1$. We write down Σ_i^2, $i = 3, 4$:

$$\begin{aligned} n_{03}\Sigma_3^2 &= \Omega_3^2(n_{03} - n_{01}\tilde{A}_2)(n_{03} - n_{02}A_2) \\ n_{03}\Sigma_4^2 &= \Omega_4^2(n_{03} - n_{01}A_1)(n_{03} - n_{02}\tilde{A}_1) \end{aligned} \tag{C.21}$$

with Ω_i^2, $i = 3, 4$, given in (C.19)–(C.20) and A_i, $\tilde{A}_i$, $i = 1, 2$, in (C.18).

C.3.3. $\boldsymbol{\Sigma_i^3 = \sum_{j=0}^3 n_{ji} = n_{3i} + \Sigma_i^2}$. To the linear n_{0i} polynomial Σ_i^2 in (C.15), we add

$$n_{3i} = \bar{n}_{3i}\Omega_2^{03}(n_{03}\bar{n}_{34} + n_{04}\bar{n}_{33} - n_{02}\,y_3 - n_{01})/\bar{n}_{34}$$

Here $\bar{n}_{3i} = n_{3i}/n_{32}$ is equal, respectively, to y_3, 1, $\bar{n}_{33}$, $\bar{n}_{34}$ for $i = 1, 2, 3, 4$. Writing the sum as a linear n_{0i} polynomial, we want to prove that the coefficient of n_{04} is the product of those for n_{01} and n_{02}. This leads for Σ_i^3 to the conditions

$$\bar{n}_{3i} = \Omega_i^2(\lambda_{4i}\bar{n}_{34} + \lambda_{3i}\bar{n}_{33} + y_3\lambda_{1i} + \lambda_{2i}), \qquad i = 1, \ldots, 4 \tag{C.22}$$

with the λ_{ji} defined in (C.16)–(C.19). If (C.22) holds, the roots of $n_{03}\Sigma_i^3$ are

$$\begin{aligned} n_{03}/n_{01} &= (\bar{n}_{3i}\Omega_2^{03}/\bar{n}_{34} - \Omega_i^2\lambda_{1i}/\lambda_{3i})/\Omega_i^3, \qquad \Omega_i^3 = \bar{n}_{3i}\Omega_2^{03} + \Omega_i^2 \\ n_{04}/n_{02} &= (\bar{n}_3\Omega_2^{03}\,y_3/\bar{n}_{34} - \Omega_i^2\lambda_{2i}/\lambda_{3i})/\Omega_i^3 \end{aligned} \tag{C.23}$$

We write down identities useful for the proof of (C.22):

$$\begin{aligned} &[4P - (P+1)S]/(S^2 - 4P) \\ &\quad = [\mu^2(1 + P - S) + 1 + P + S]/[\mu^2(S - P - 1) + 1 + P + S] \\ &\qquad (\mu - 1)P\lambda_{41} - (\mu + 1)\lambda_{31} = [4P - S(P+1)]E_1E_2 \\ &(E_1E_2)^{-1} = \{[4P - S(P+1)]S \\ &\qquad + \mu S^2(P-1)(S-P-1)/(S+P+1)\}/8P(S^2 - 4P) \end{aligned} \tag{C.24}$$

These identities depend upon μ, P, and S, which are considered as independent variables. For their proofs we identify both sides of the relations, substituting μ^2 by (C.5).

a. Σ_i^3, $i=1, 2$. For Σ_1^3 the lhs of (C.22) is y_3 and we rewrite

$$2y_3[4P-S(P+1)]-\lambda_{21}=\bar{n}_{34}\lambda_{41}+\bar{n}_{33}\lambda_{31} \qquad \text{(C.22')}$$

From (C.5) for $\bar{n}_{33}$, $\bar{n}_{34}$ we see that the rhs is linear in y_3. Further, y_3 is only in the first term of the lhs. We identify terms proportional to or independent of y_3. In the rhs the term proportional to y_3 is $2[(\mu-1)P\lambda_{41}-(\mu+1)\lambda_{31}]/D_1D_2$ and with the identity (C.24) it is equal to the y_3 term of the lhs. The y_3-independent term in the rhs is $2/C_1C_2$ multiplied by the factor $-(\mu+1)P\lambda_{41}+(\mu-1)\lambda_{31}$. With the identity (C.24) this factor becomes $(S^2-4P)E_1E_2$, so that the y_3-independent term is $-\lambda_{21}$. For Σ_2^3, we start with Σ_1^3 and use the transform $(n_{01}\leftrightarrow n_{02},\ y_3\to y_3^{-1},\ \ \bar{n}_{3i}\to\bar{n}_{3i}y_3^{-1})$, while the roots are obtained from (C.23):

$$\begin{aligned}
n_{03}\Sigma_1^3 &= \Omega_1^3(n_{03}-n_{01}A_5)(n_{03}-n_{02}A_6)\\
n_{03}\Sigma_2^3 &= \Omega_2^3(n_{03}-n_{01}\tilde{A}_6)(n_{03}-n_{02}\tilde{A}_5)\\
A_5 &= (\Omega_1^{03}/\bar{n}_{34}+A_1\Omega_1^2)/\Omega_1^3\\
A_6 &= (\Omega_1^{03}\,y_3/\bar{n}_{34}+A_2\Omega_1^2)/\Omega_1^3\\
\Omega_i^3 &= \Omega_i^2+\Omega_i^{03}\\
\tilde{A}_5 &= (\Omega_2^{03}\,y_3/\bar{n}_{34}+\Omega_2^2\tilde{A}_1)/\Omega_2^3\\
\tilde{A}_6 &= (\Omega_2^{03}/\bar{n}_{34}+\Omega_2^2\tilde{A}_2)/\Omega_2^3
\end{aligned} \qquad \text{(C.25)}$$

b. Σ_i^3, $i=3, 4$. For Σ_3^3, the lhs of (C.22) is $\bar{n}_{33}$ and we rewrite

$$2(4P-S^2)(\bar{n}_{33}+\bar{n}_{34}/P)=y_3\lambda_{13}+\lambda_{23}=y_3\lambda_{23}(\mu\to-\mu)+\lambda_{23} \qquad \text{(C.22'')}$$

With (C.5) for $\bar{n}_{33}$, $\bar{n}_{34}$, the lhs has a structure similar to the rhs: $y_3H(-\mu)+H(\mu)$ with

$$H(-\mu)=4(4P-S^2)[\mu(S-P-1)-S-P-1]/SE_1E_2$$

and we must verify that $H(\mu)=\lambda_{23}$. Using the third identity (C.24), we find

$$2PH(\mu)=\frac{[\mu(1-P)\,S(S-P-1)/(S+P+1)-4P+S(P+1)]}{[\mu(S-P-1)-S-P-1)]^{-1}}$$

In the product we use (C.3) for μ^2 and identify with λ_{23}. For Σ_4^3 we exchange $3\leftrightarrow 4$ and finally obtain

$$n_{03}\Sigma_3^3 = (n_{03} - A_5 n_{01})(n_{03} - \tilde{A}_5 n_{02})\Omega_3^3$$
$$n_{03}\Sigma_4^3 = (n_{03} - n_{01}\tilde{A}_6)(n_{03} - n_{02}A_6) \tag{C.26}$$
$$\Omega_3^3 = \bar{n}_{33}\Omega_2^{03} + \Omega_3^2, \qquad \Omega_4^3 = \Omega_4^{03} + \Omega_4^2$$

C.4. Σ_i for $S = -2(P+1)$ and $\mu = (1-P)/3(1+P)$

C.4.1. Calculations of the Σ_i. It is useful to introduce a new parameter Q, a function of P; from (C.5)–(C.6) we find for the cubic y_3 equation and $\bar{n}_{33}, \bar{n}_{34}$:

$$Q = 3P/(1+P+P^2), \qquad y_3^3 + Q^2 + y_3(y_3+Q)(4+Q) = 0$$
$$3P\bar{n}_{33} = Q(2P+1) + y_3(P+2), \qquad 3\bar{n}_{34} = Q(P+2) + y_3(2P+1) \tag{C.27}$$

For Σ_i^{03} the expressions of A_1, A_2, and Ω_i^{03} in terms of y_3, $\bar{n}_{33}$, and $\bar{n}_{34}$ are the same as (C.11). For Σ_i^2, due to the use of the transform $\mu \to -\mu$, we write down some parameters as functions of μ, P:

$$C_1C_2 = -2(1+P^2+4P)/3(P+1) + 2\mu(P-1) = 4P/Q(P+1)$$
$$E_1E_2 = C_1C_2(\mu \to -\mu) = -4P/(P+1)$$
$$\lambda_{21} = -8(P^2+P+1)E_1E_2/C_1C_2 \tag{C.28}$$
$$\lambda_{31} = -6\mu(P+1)(2P+1) + 8P^2 + 2P + 2$$
$$\Omega_1^2(2P+1)(P+2) = \lambda_{31}/2$$

which lead with our chice for the square root of $\mu^2 = [(1-P)/3(1+P)]^2$ to

$$A_1 = 1/2P, \qquad A_2 = 2/P, \qquad \tilde{A}_1 = Q/2, \qquad \tilde{A}_2 = 2/Q$$
$$\Omega_1^2 = 6P^2/(2P+1)(P+2) = 2P\Omega_3^2 \tag{C.29}$$
$$\Omega_2^2 = 6P/Q(P+2)(2P+1) = \Omega_4^2/2P$$

For Σ_3^3 we make explicit the expressions (C.25) and write down Ω_2^{03} for the Ω_i^3, $i = 3, 4$ [see (C.26)]

$$A_5 = (Q^2 + y_3^2 + Qy_3 - 3y_3)/(2Q + y_3)(PQ - y_3)$$
$$A_6 = (2Q + y_3)/(PQ - y_3)$$
$$\tilde{A}_5 = Q(Q^2 + y_3^2 + Qy_3 - 3y_3)/(2y_3 + Q)(y_3 - QP)$$
$$\tilde{A}_6 = (2y_3 + Q)/(Q(y_3 - PQ) \tag{C.30}$$
$$\text{def: } X = 9P(\bar{n}_{33}\bar{n}_{34} - y_3) = (P+2)(2P+1)(Q^2 + y_3^2 + Qy_3)$$
$$X\Omega_1^3 = 3P(QP - y_3)(2Q + y_3)$$
$$XQ\Omega_2^3 = -3P(2y_3 + Q)(PQ - y_3), \qquad X\Omega_2^{03} = -\bar{n}_{34}9P$$

C.4.2. Sufficient Positivity Conditions for Σ_i. Let us choose $0<P<1$; then the cubic equation (C.27) with $0<Q<1$ has three real roots. We are interested in the y_3 root such that $-1<y_3<0$, which, of course, cannot be written down explicitly in terms of Q. However, this determination can be defined by appropriate choices of two associated quadratic equations:

$$\begin{aligned} P\beta &= 1-(1-\beta^2)^{1/2}, \qquad \beta = 2Q/(3-Q) \\ Q &= -2y_3[1-(1-\alpha)^{1/2}]/\alpha, \qquad \alpha = 4(1+y_3)/(4+y_3) \end{aligned} \tag{C.31}$$

from which we easily find the following results.

Lemma 6. If $-1<y_3<0$, then $0<\alpha<1$, $0<Q<1$, $0<\beta<1$, and $0<P<1$.

In the sequel we always assume the y_3 determination such that

$$0<P<1, \qquad -1<y_3<0, \qquad 0<Q<1 \tag{C.32}$$

and for $\Sigma_i>0$, we seek conditions on P, n_{01}, n_{03}.

a. Positivity for Σ_i^2. Since the roots are positive, we must check the signs of n_{03}^2 and the locations of the roots.

Lemma 7. $A_1<A_2$, $A_1<\tilde{A}_2$, $\tilde{A}_1<A_2$, $\Omega_i^2>0$.

$A_1<A_2$ is obvious from (C.29); $A_1<\tilde{A}_2$ is equivalent to $0<1+4(P+P^2)$; $A_1<\tilde{A}_2$ is equivalent to $0<4(1+P)+P^2$; and Ω_i^2 as well as P, Q are positive.

Lemma 8. $\Sigma_i^2>0$ if $n_{01}<n_{02}\tilde{A}_1/A_1=n_{02}PQ$, $0<n_{03}<n_{01}A_1$.

Due to $PQ<1$, we get $n_{01}<n_{02}$ and

$$n_{01}A_1 = \inf(n_{01}A_1, n_{01}\tilde{A}_2, n_{02}\tilde{A}_1, n_{02}A_2)$$

Since the coefficients of n_{03}^2 are positive and n_{03} is outside the four intervals constituted by the roots, then $\Sigma_i^2>0$.

b. Positivity for Σ_i^{03}. Here two roots are positive (A_3, $\tilde{A}_4$ positive), while the two other are negative ($\tilde{A}_3$, A_4). Similarly, the coefficients of n_{03}^2 for $i=1,3$ are positive, while those for $i=2,4$ are negative. For these results we must first find inequalities for y_3/Q in (C.31).

Lemma 9. $-1<y_3/Q<-1/2$ and $y_3/Q+(2P+1)/(P+2)<0$.

From (C.31) for $Q=Q(y_3,\alpha)$ we have both the inequalities $(1-\alpha)^{1/2}<1-\alpha/2$ and $>1-\alpha$ and the first two inequalities of the lemma follow. For the last inequality we define a scaling parameter $\bar{y}$ and from the cubic equation (C.27) find $Q=Q(\bar{y})$:

$$\bar{y}=y_3/Q \rightarrow Q=-(2\bar{y}+1)^2/(\bar{y}+\bar{y}^2+\bar{y}^3), \qquad -1<\bar{y}<-1/2 \tag{C.33}$$

From (C.31) for $P=P(\beta)$ and $(1-\beta^2)^{1/2}>1-\beta^2$ we find the bound $P<2Q/(3-Q)$. Further, since $(2P+1)/(P+2)$ is increasing, it is bounded by the expression obtained by substituting the Q (or $\bar{y}$) dependent bound of P:

$$\bar{y}+(2P+1)/(P+2)<(1+2\bar{y})(\bar{y}+1)(1-\bar{y}^2)/2(\bar{y}+\bar{y}^2+\bar{y}^3)<0$$

Lemma 10. $\bar{n}_{33}<0$, $\bar{n}_{34}>0$.

From (C.27) for $\bar{n}_{3i}$ and Lemma 9 we find

$$3P\bar{n}_{33}/Q=2P+1+(P+2)\,y_3/Q<0$$

$$3\bar{n}_{34}/Q=P+2+(2P+1)\,y_3/Q>1-P>0$$

Lemma 11. $\bar{n}_{33}\bar{n}_{34}-y_3>0$ and $\Omega_i^{03}>0$ for $i=1,3$ (<0 for $i=2,4$), and $A_3>0$, $A_4<0$, $\tilde{A}_3<0$, and $\tilde{A}_4>0$.

From the explicit expression of X given in (C.30), we see both the first inequality and $\Omega_2^{03}<0$. From the expressions (C.11) linking Ω_2^{03} and the other Ω_i^{03} and Lemma 10 the signs of Ω_i^{03} follow. The signs for the roots A_k, $\tilde{A}_k$, $k=3,4$, given in (C.11) are consequences of the signs of y_3, $\bar{n}_{33}$, $\bar{n}_{34}$.

Lemma 12. $0<A_3<\tilde{A}_4$.

We find $\tilde{A}_4-A_3=(y_3-\bar{n}_{33}\bar{n}_{34})/y_3\bar{n}_{34}$ and apply Lemmas 10 and 11.

Lemma 13. $\Sigma_i^{03}>0$ if $0<n_{01}A_3<n_{03}<n_{01}\tilde{A}_4$.

For each Σ_i^{03} one root is positive, while the other is negative. From the signs of the coefficients of n_{03}^2 and of the roots we obtain $\Sigma_i^{03}>0$, $i=1,3$, if $n_{03}>n_{01}A_3$; $\Sigma_i^{03}>0$, $i=2,4$, if $0<n_{03}<n_{01}\tilde{A}_4$. On the other hand, due to Lemma 12, the interval $(n_{01}A_3,\ n_{01}A_4)$ is not empty and n_{03} must stay inside this interval.

c. Positivity for Σ_i^3. Here all the roots as well as the coefficients of n_{03}^3 are positive.

Lemma 14. $\Omega_i^3>0$, $i=1,2,3$.

Due to $X>0$ in (C.30), the sign of Ω_1^3 is that of $2Q+y_3>Q+y_3>0$ from Lemma 9. With this lemma the sign of Ω_2^3, given by $-(2y_3+Q)$, is positive. Finally, Ω_3^3 written down in (C.26) is the sum of two positive terms.

Lemma 15. $\bar{n}_{34}<Q/2$, $4P/Q>1$, and $\Omega_4^3>0$.

We have $3\bar{n}_{34}=Q(P+2)+y_3(2P+1)$ and we find the first inequality. For the second we get $4P/Q=4(1+P+P^2)/3>1$. Let us rewrite $\Omega_4^3=\Omega_4^{03}+\Omega_4^2$ and apply these results:

$$\Omega_4^3=\bar{n}_{34}(\Omega_2^{03}+2P\Omega_2^2/\bar{n}_{34})>\bar{n}_{34}(\Omega_2^{03}+4P\Omega_2^2/Q)>\bar{n}_{34}\Omega_2^3>0$$

Lemma 16. $A_k > 0$, $\tilde{A}_k > 0$, $k = 5, 6$.

These results follow from the explicit expressions (C.30) and from the above inequalities: $y_3 < 0$, $2y_3 + Q < 0$, and $y_3 + Q > 0$.

The roots and the signs of n_{03}^2 are positive, so the Σ_i^3 will be positive for n_{03} less than the smallest root and we must compare the A_k, $\tilde{A}_k$.

Lemma 17. $\tilde{A}_6 < A_5$, $A_6 < \tilde{A}_5$, and $PQ\tilde{A}_6 < A_6$.

Using Lemma 9 and the expressions written down in (C.30), we find

$$\tilde{A}_6/A_5 = A_6/\tilde{A}_5 = -(2y_3 + Q)(2Q + y_3)/Q(Q^2 + y_3^2 + Qy_3 - 3y_3)$$

$$\begin{aligned} 1 - A_6/A_5 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (C.34) \\ &= [Q(Q^2 + y_3^2 + y_3 Q) + 2(y_3 + Q + y_3^2)]/Q(Q^2 + y_3^2 + Qy_3 - 3y_3) > 0 \\ PQ\tilde{A}_6 - A_6 &= [(P+1)(Q + y_3) + Q + Py_3]/(y_3 - PQ) < 0 \end{aligned}$$

We notice that $Q + Py_3 > Q + y_3 > 0$.

Lemma 18. $\Sigma_i^3 > 0$ if $0 < n_{03} < n_{01} A_6$ and $n_{01} < n_{02} PQ$.

It is sufficient to prove that $n_{01}\tilde{A}_6$ is the smallest among the four roots of Σ_i^3. From Lemma 17 and the assumptions of Lemma 18 we find $n_{01}\tilde{A}_6 < n_{02} PQ\tilde{A}_6 < n_{02} A_6 < n_{02}\tilde{A}_5$ and $n_{01}\tilde{A}_6 < n_{01} A_5$.

d. Positivity for all Σ_i. For the positivity of Σ_i^2, Σ_i^3, Σ_i^{03} separately we have found three n_{03} intervals. It remains to show that their intersections is not empty. We want to prove that the interval $(n_{01}A_3, n_{01}\tilde{A}_6)$ is the intersection of $(0, n_{01}A_1)$, $(0, n_{01}\tilde{A}_6)$, and $(n_{01}A_3, n_{01}\tilde{A}_4)$.

Lemma 19. $\tilde{A}_6 < A_1$ and $A_3 < \tilde{A}_6 < \tilde{A}_4$.

These results come from the explicit expressions

$$\begin{aligned} \tilde{A}_6 - A_1 &= \bar{n}_{34}(2P+1)/2P(y_3 - PQ) < 0 \\ A_3 - \tilde{A}_6 &= (2P+1)(Q^2 + y_3^2 + Qy_3)/3y_3(PQ - y_3) < 0 \qquad (C.35) \\ \tilde{A}_6 - \tilde{A}_4 &= 6(P+1)(Q^2 + y_3^2 + Qy_3)/Q(y_3 - PQ)\bar{n}_{34} < 0 \end{aligned}$$

Theorem 2. Sufficient conditions in order to have all 12 $\Sigma_i > 0$ are

$$0 < P < 1 \quad (-1 < y_3 < 0), \qquad 0 < n_{01} < n_{02} PQ, \qquad n_{01} A_3 < n_{03} < n_{01}\tilde{A}_6$$

with $A_3 = \bar{n}_{33}/y_3$ and Q, y_3, $\bar{n}_{33}$ functions of P given in (C.27), while $\tilde{A}_6$ is written down in (C.30). Finally, we write $z_j = z_+$ such that the product is P and the sum $-2(P+1)$,

$$z_{\pm} = -P - 1 \pm (P^2 + P + 1)^{1/2}, \qquad z_+ + 1 > 0, \qquad z_- + 1 < 0 \qquad (C.36)$$

C.4.3. $\tau_1\tau_2>0$. The sign of $\tau_1\tau_2$ is given by the product of two quadratic n_{03} polynomials:

$$\tau_1\tau_2 4n_{03}^2/3(P+1)=\mathscr{T}_1\mathscr{T}_2, \qquad \mathscr{T}_i=n_{03}^2-2n_{03}\alpha_\pm+n_{01}n_{02}/z_\pm$$

$$\operatorname{def}\alpha_\pm=n_{01}/E_\pm+n_{02}/C_\pm, \qquad C_\pm=-2/[1\pm(P+2)/(P^2+P+1)^{1/2}]$$

$$E_\pm=-6P/[2P^2+2P-1\pm(2P+1)(P^2+P+1)^{1/2}]$$

with $E_\pm, C_\pm$ the quantities E_i, C_i for $z_i=z_\pm$ defined in (C.2) for the general formalism and calculated here for $S=-2(P+1)$ and $3\mu=(1-P)/(1+P)$. For each i value the two roots of the polynomial $\mathscr{T}_i$ are real and opposite ($z_\pm<0$). It follows that $\tau_1\tau_2>0$ if, for instance, $0<n_{03}<\inf(n_{03,z_+}, n_{03,z_-})$, where the two positive roots are

$$n_{03,z_\pm}=\alpha_\pm+\sqrt{\Delta_\pm}, \qquad \Delta_\pm=\alpha_\pm^2-n_{01}n_{02}/z_\pm \tag{C.38}$$

From Theorem 2 we must have $n_{03}<n_{01}\tilde{A}_6<n_{01}A_1$ (see Lemma 19). Then a sufficient condition is

$$\tau_1\tau_2>0 \qquad \text{if} \quad n_{03,z_\pm}>n_{01}A_1=n_{01}/2P \tag{C.39}$$

Lemma 20. $C_+<0$, $E_+<0$, $\alpha_+<0$; and $C_->0$, $E_->0$, $\alpha_->0$. These are consequences of the assumption (C.32) for P and $n_{0i}>0$.

Lemma 21. def $X_+=n_{02}/z_++n_{01}/4P^2-\alpha_+/P$; then $X_+<0$ and $n_{03,z_+}>n_{01}/2P$.

We have

$$X_+=n_{02}(1/z_+-1/PC_+)+n_{01}(1/4P^2-1/PE_+)$$

Since the coefficient of n_{01} is positive and $n_{01}<n_{02}PQ$ it follows that

$$\begin{aligned} X_+/n_{02}&<1/z_+-1/PC_++Q/4P-Q/E_+\\ &=-(P^2+5P/2+1)/2(P^3+P^2+P)<0 \end{aligned}$$

Consequently, we get $X_+n_{01}+\alpha_+^2<\alpha_+^2$ or

$$(n_{01}/2P-\alpha_+)^2<\alpha_+^2-n_{01}n_{02}/z_+=\Delta_+^2$$

Taking the positive square-root determination in both sides of the inequality, we get $\Delta_+^{1/2}+\alpha_+=n_{03,z_+}>n_{01}/2P$.

Lemma 22. def $X_-=n_{01}/2P-\alpha_-$; then $X_-<0$ and $n_{03,z_-}>n_{01}/2P$.

We have $X_- = n_{01}(1/2P - 1/E_-) - n_{02}/C_-$. The coefficient of n_{01} is still positive for our solutions with $\Sigma_i > 0$, n_{01} sup $< n_{02}PQ$, then

$$X_-/n_{02} < -1/C_- + Q/2 - QP/E_- = (2P+1)/2 - (P^2+P+1)^{1/2} < 0$$

Consequently, we get $n_{01}/2P < \alpha_- < \alpha_- + \Delta_-^{1/2} = n_{03, z_-}$.

Theorem 2bis. The sufficient conditions of Theorem 2 lead to N_i solutions with $\tau_1 \tau_2 > 0$; then, for these solutions, their asymptotic positivity conditions $\Sigma_i > 0$ are satisfied.

C.4.4. Another n_{03} Interval Leading to $\Sigma_i > 0$. For Σ_i^2, Σ_i^3 all coefficients of n_{03}^2 as well as all roots are positive. Instead of the n_{03} interval less than the smallest root as in Theorem 2, we choose the n_{03} interval larger than the highest root and the two Σ_i will be positive. Further, if this highest root belongs to the interval $(n_{01}A_3, n_{01}\tilde{A}_4)$, then $\Sigma_i^{03} > 0$ with $n_{01}A_3$ replaced by the highest root. We still assume $0 < P < 1$ and $-1 < y_3 < 0$.

Lemma 23. If $n_{01}/n_{02} > A_2/\tilde{A}_2 = Q/P > 1$ and if $n_{03}/n_{01} > \sup(\tilde{A}_2, A_5)$, then $\Sigma_i^2 > 0$, $\Sigma_i^3 > 0$.

Due to the assumption and Lemma 7, we find

$$n_{01}\tilde{A}_2 = \sup(n_{01}A_1, n_{02}A_2, n_{01}\tilde{A}_2, n_{02}\tilde{A}_1)$$

and $\Sigma_i^2 > 0$. Further from the relation

$$A_2/\tilde{A}_2 - \tilde{A}_5/A_5 = 3\bar{n}_{33}/(2y_3 + Q) > 0$$

we see that

$$n_{01}A_5/n_{02}\tilde{A}_5 > A_2 A_5/\tilde{A}_2\tilde{A}_5 > 1$$

Adding the results of Lemma 17, we get

$$n_{01}A_5 = \sup(n_{01}A_5, n_{02}A_6, n_{01}\tilde{A}_6, n_{02}\tilde{A}_5)$$

and $\Sigma_i^3 > 0$.

Lemma 24. $A_3 < \tilde{A}_2 < \tilde{A}_4$, $A_5 < \tilde{A}_4$, and $\Sigma_i^{03} > 0$ for $\sup(\tilde{A}_2, A_5) < n_{03}/n_{01} < \tilde{A}_4$.

These results are deduced from the identities

$$\tilde{A}_2/\tilde{A}_4 - 1 = (2P+1)(2y_3 + Q)/3 < 0$$

$$A_3/\tilde{A}_2 - 1 = (2P+1)\,Q(2Q/3Py_3 - 1/6) < 0$$

$$A_5/\tilde{A}_4 - 1 = (Q^2 + y_3^2 + y_3 Q)(2P+1)(y_3 - Q)/3(2Q + y_3)(QP - y_3) < 0$$

and applying the previous results: $y_3 < 0$, $2y_3 + Q < 0$, $Q + y_3 > 0$. We obtain the following theorem.

Theorem 3. The Σ_i are positive if P and the n_{0i} are chosen such that:

$$0 < P < 1 \qquad (-1 < y_3 < 0), \qquad n_{01} > n_{02} Q/P, \\ \sup(\tilde{A}_2, A_5) < n_{03}/n_{01} < \tilde{A}_4 \tag{C.40}$$

We notice that numerically we have found $\tilde{A}_2 < A_5$.

The problem of $\tau_1 \tau_2 > 0$ remains as in Section C.4.3. This property holds if $n_{03} > \sup(n_{03,z_+}, n_{03,z_-})$, which gives for the allowed interval the sufficient condition

$$\tau_1 \tau_2 > 0 \qquad \text{if} \quad n_{03,z_\pm} < n_{01} \tilde{A}_2 = n_{01}\, 2/Q \tag{C.41}$$

Lemma 25. def $X_+ = n_{01}/z_+ + 4n_{01}/Q^2 - 4\alpha_+/Q > 0$ and $n_{03,z_+} < 2n_{01}/Q$.

α_+, z_+, C_+, E_+, and α_-,... are written down in (C.36)–(C.38). We have

$$X_+ = n_{01}(4/Q^2 - 4/QE_+) + n_{02}(1/z_+ - 4/QC_+)$$

Due to $E_+ < 0$, the coefficient of n_{01} is positive; for n_{02} we find $(2P+1)[P - 1 + (1 + P + P^2)^{1/2}]/3P$ and $X_+ > 0$. Then we get $X_+ n_{01} + \alpha_+^2 > \alpha_+^2$ or $(n_{01} 2/Q - \alpha_+)^2 > \Delta_+$. Taking the positive square-root determination in both sides, $n_{01} 2/Q > \Delta_+^{1/2} + \alpha_+ = n_{03,z_+}$.

Lemma 26. def $X_- = n_{01}/z_- + 4n_{01}/Q^2 - 4\alpha_-/Q > 0$ and $n_{03,z_-} < 2n_{01}/Q$.

We have

$$X_- = n_{01}(4/Q^2 - 4/QE_-) + n_{02}(1/z_- - 4/QC)$$

The coefficient of n_{01} is positive:

$$1/Q - 1/E_- = [3 + (2P+1)(1 + P + P^2)^{1/2}]/6P$$

while the coefficient of n_{02} is negative:

$$1/z_- - 4/QC_- = (2P+1)[P - 1 - (1 + P + P^2)^{1/2}]3P$$

X_- is positive if it is positive for $\sup n_{02} = n_{01} P/Q$. We find

$$X_-/n_{01} > [6 - P - P^2 + 2P^3 + (2P+1)(2-P)(1 + P + P^2)^{1/2}]/3PQ > 0$$

From $X_- n_{01} + \alpha_-^2 > \alpha_-^2$ or $(n_{01} 2/Q - \alpha_-)^2 > \Delta_-$. With similar calculations as above we find $n_{01} 2/Q - \alpha_- = n_{01}(2/Q - 1/E_-) - n_{02}/C_-$; the coefficient of n_{01} is positive while the coefficient of n_{02} is negative. However, for $\sup n_{02}$ the sum is still positive. Consequently, taking the positive square root in both sides of the last inequality, we find $n_{01}/2Q - \alpha_{-}\Delta_{-}^{1/2}$ or $n_{03,z_{-}} < 2n_{01}/Q$.

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